**6.4**

**Q14. Prove the identity**

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**whenever n, r, and k are nonnegative integers with r ≤ n and k ≤ r**

**a) using a combinatorial argument.**

Suppose that we have a set with n elements, and we wish to choose a subset A with k elements and

another, disjoint, subset with r − k elements. The left-hand side gives us the number of ways to do this,

namely the product of the number of ways to choose the r elements that are to go into one or the other of the subsets and the number of ways to choose which of these elements are to go into the first of the subsets. The 166 Chapter 6 Counting right-hand side gives us the number of ways to do this as well, namely the product of the number of ways to choose the first subset and the number of ways to choose the second subset from the elements that remain.

**b) using an argument based on the formula for the number of r-combinations of a set with n elements.**

On the one hand,

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and on the other hand

**Q23. In this exercise we will count the number of paths in the xy plane between the origin (0, 0) and point (m, n), where m and n are nonnegative integers, such that each path is**

**made up of a series of steps, where each step is amove one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from**

**(0, 0) to (5, 3) are illustrated here.**

**a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.**

A path of the desired type consists of m moves to the right and n moves up. Each such path can be represented by a bit string of length m + n with m 0s and n 1s, where a 0 represents a move to the right and a 1 a move up

**b) Conclude from part (a) that there are paths of the desired type.**

The number of bit strings of length m + n containing exactly n 1s equals  = because such a string is determined by specifying the positions of the n 1s or by specifying the positions of the m 0s

**Q26. Use Exercise 23 to prove Pascal’s identity. [Hint: Show that a path of the type described in Exercise 33 from (0, 0) to (n + 1 − k, k) passes through either (n + 1 − k, k − 1) or (n − k, k), but not through both.**

A path ending up at (n + 1 − k, k) must have made its last step either upward or to the right. If the last

step was made upward, then it came from (n + 1 − k, k − 1); if it was made to the right, then it came from

(n − k, k). The path cannot have passed through both of these points. Therefore the number of paths to

(n+1−k, k) is the sum of the number of paths to (n+1−k, k−1) and the number of paths to (n−k, k). By

Exercise 33 this tells us that

which simplifies to

Pascal’s identity