King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #3
(Due Wednesday August 9, 2017 at midnight)

I. **Solve and submit the solutions to Questions 1-9. In all questions below, show your intermediate work. Make sure to attempt these questions without using a calculator.**

1) (10 points) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Assume that the product of a rational number a & an irrational number q is rational.

i.e., $aq = r$ where r is rational.

Since $a \neq 0$, $q = \frac{r}{a}$.

Since $r, a \in \mathbb{Q}$, then $\frac{r}{a} \in \mathbb{Q}$ (rational)

* contradiction *.

$\therefore$ r is irrational.

$\langle 10, 9, 7, 5, 2, 0 \rangle$

2) (10 points) Prove that, if both solutions of $x^2 + ax + b = 0$ are even integers, then a and b are both even integers.

Let $r_1, r_2 \in \mathbb{Z}$ even integers that are solutions to the equation.

$r_1 = 2n_1 \quad n_1 \in \mathbb{Z}.$

$r_2 = 2n_2 \quad n_2 \in \mathbb{Z}.$

+3

$(x - r_1)(x - r_2) = x^2 + ax + b$

<7,6,4,2>

$(x - 2n_1)(x - 2n_2) = x^2 + ax + b$

$x^2 - 2n_1 x - 2n_2 x + 4n_1 n_2 = x^2 + ax + b$

$\therefore \quad a = -2n_1 - 2n_2 = 2(-n_1 - n_2)$ even

$b = 4n_1 n_2 = 2(2n_1 n_2)$ even.

3) (10 points) Let $A_n = \left[-\frac{1}{i}, \frac{1}{i}\right]$. Find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$.

$$\bigcup_{i=1}^{n} A_i = [-1,1] \cup [-\tfrac{1}{2}, \tfrac{1}{2}] \cup \dots \cup [-\tfrac{1}{n}, \tfrac{1}{n}] = [-1,1] = A_1.$$

<5,2,0>

$$\bigcap_{i=1}^{n} A_i = [-1,1] \cap [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \dots \cap [-\tfrac{1}{n}, \tfrac{1}{n}] = [-\tfrac{1}{n}, \tfrac{1}{n}] = A_n.$$

<5,2,0>

4) (10 points) Can you conclude that $A = B$ if A, B, and C are sets such that $A \cup C = B \cup C$ and $A \cap C = B \cap C$? Prove your answer.

Using Membership table.

<10,9,7,4,2,0>

A	B	C	$A\cup C$	$B\cup C$	$A\cap C$	$B\cap C$	$A\cup C=B\cup C$	$A\cap C=B\cap C$
0	0	0	0	0	0	0	1	1
0	0	1	1	1	0	0	1	1
0	1	0	0	1	0	0	0	1
0	1	1	1	1	0	1	1	0
1	0	0	1	0	0	0	0	1
1	0	1	1	1	1	0	1	0
1	1	0	1	1	0	0	1	1
1	1	1	1	1	1	1	1	1

Since $A = B \iff (A\cup C = B\cup C)$ and $(A\cap C = B\cap C)$

$\therefore$ if $(A\cup C = B\cup C) \wedge (A\cap C = B\cap C)$ then $A = B$.

5) (10 points) Determine whether the function $f(m,n) = m^3 - n^3$ where $f: Z \times Z \to Z$ is onto. Prove your answer.

+6

Not onto. $f(m,n) = (m-n)(m^2+mn+n^2)$

Now, $f(m,n) = 5$ if and only if $|m-n| = 1$ and $|m^2+mn+n^2| = 5$ or $|m-n| = 5$ and $|m^2+mn+n^2| = 1$.

By inspection, no $m,n \in \mathbb{Z}$ s.t. $m^2+mn+n^2 = 5$ with $|m-n| = 1$ & vise versa.

<4, 2, 0>

If they say onto with justification (although wrong) give +3

Hence, f is not onto.

6) (10 points) Find $f \circ g$ and $g \circ f$ where $f(x) = x^3 - 5$ and $g(x) = 2x + 3$ where f and g are functions from $\mathbb{R}$ to $\mathbb{R}$.

$$f \circ g(x) = f(2x+3) = (2x+3)^3 - 5$$

<5, 3, 1>

$$= 8x^3 + 3(4x^2(3)) + 3(2x)(9) + 27 - 5$$

$$= 8x^3 + 36x^2 + 54x + 22$$

$$g \circ f(x) = g(x^3-5) = 2(x^3-5) + 3$$

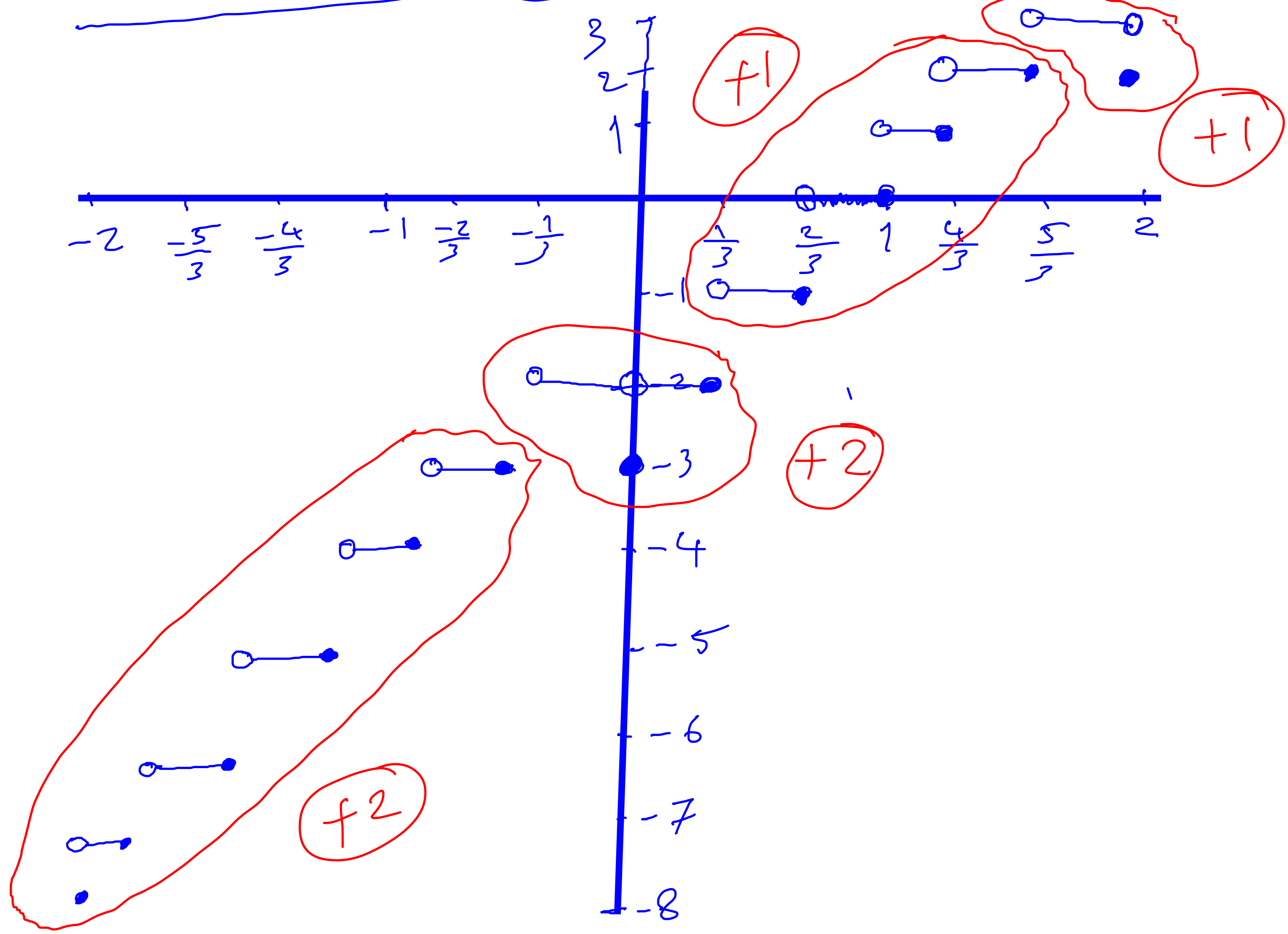
$$= 2x^3 - 7$$

<5, 3, 1>

+4

7) (10 points) Draw the graph of $f(x) = \lceil 3x - 1 \rceil - \left\lfloor \frac{x}{2} + 2 \right\rfloor$ where $x \in [-2,2]$.

$\lceil 3x-1 \rceil = -7$ $-8 < 3x-1 \le -7$ $-7 < 3x \le -6$ $-\frac{7}{3} < x \le -2$	$\lceil 3x-1 \rceil = -6$ $-7 < 3x-1 \le -6$ $-6 < 3x \le -5$ $-2 < x \le -\frac{5}{3}$	$\lceil 3x-1 \rceil = -5$ $-6 < 3x-1 \le -5$ $-5 < 3x \le -4$ $-\frac{5}{3} < x \le -\frac{4}{3}$, ... etc.
$\lfloor \frac{x}{2}+2 \rfloor = 1$ $1 \le \frac{x}{2}+2 < 2$ $-1 \le \frac{x}{2} < 0$ $-2 \le x < 0$	$\lfloor \frac{x}{2}+2 \rfloor = 2$ $2 \le \frac{x}{2}+2 < 3$ $0 \le \frac{x}{2} < 1$ $0 \le x < 2$	.... etc.



8) (10 points) Prove that

$$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^{j} = \frac{2^{n+1}+(-1)^{n}}{3\times 2^{n}}$$

Proof by Mathematical Induction:

$\langle 3, 1 \rangle$

1. Basis: $n=0$: $\sum_{j=0}^{0}\left(-\frac{1}{2}\right)^{j} = 1 \overset{?}{=} \frac{2^{1}+(-1)^{0}}{3\times 2^{0}}$

$= \frac{3}{3} = 1$ ✓

$\langle 7, 5, 3 \rangle$

2. Inductive step: Assume that $P(k)$ holds for $0 \le k \le n$. To show that $P(n+1)$ holds.

$$\sum_{j=0}^{n+1}\left(-\frac{1}{2}\right)^{j} = \sum_{j=0}^{n}\left(-\frac{1}{2}\right)^{j} + \left(-\frac{1}{2}\right)^{n+1}$$

$$= \frac{2^{n+1}+(-1)^{n}}{3\times 2^{n}} + \frac{(-1)^{n+1}}{2^{n+1}}$$ By Induction hypothesis

$$= \frac{2\cdot 2^{n+1} + (-1)^{n}\cdot 2 + 3(-1)^{n+1}}{3\times 2^{n+1}}$$

$$= \frac{2^{n+2} - 2(-1)^{n+1} + 3(-1)^{n+1}}{3\times 2^{n+1}}$$

$$= \frac{2^{n+2} + (-1)^{n+1}}{3\times 2^{n+1}}$$

9) (10 points) What amounts of postage can be assembled using 2-cent and 5-cent stamps only? Prove your answer using strong induction.

All values k s.t. $k \geq 4$. +2

Basis: $4 = 2(2) + 0(5) \to (2,0)$ +2

$5 = 0(2) + 1(5) \to (0,1)$ +2

Inductive step: Assume the result holds for all k, $5 \leq k \leq n$.

For $n+1$, since $4 \leq n-1 \leq n$ & using induction hypothesis, $\exists\ i, j \in \mathbb{Z}^+ \cup \{0\}$ s.t.

$\langle 4, 2, 0 \rangle$

$2i + 5j = n-1$

adding 1 2-cent stamp,

$2i + 5j + 2 = n+1$

$2(i+1) + 5j = n+1$

$\therefore (i+1, j)$ produces $n+1$.