King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #4
(Due Friday August 10, 2018 at midnight)

I. Solve and submit the solutions to Questions 1-8. In all questions below, show your intermediate work. Make sure to attempt these questions without using a calculator.

1) (10 points) How many subsets of a set with 100 elements have more than two elements?

Answer: Total # of subsets − (# subsets with 0 elts +
" " 1 elt +
" " 2 elts)

Total # of subsets $= 2^{100}$

# of subsets with 0 elts $= \binom{100}{0} = 1$

# of subsets with 1 elt $= \binom{100}{1} = 100$

# of subsets with 2 elts $= \binom{100}{2} = \frac{100(99)}{2}$

$= 50(99)$.

$\therefore$ Answer : $2^{100} - (1 + 100 + 50(99))$

2) (5 points) A computer network consists of 10 computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

Since every computer is connected to at least one other computer, in the worst case, we will have each computer connected to i other computers, where i ranges between 1 and 9. Since the 10 computers are distributed among the 9 numbers 1..9, the pigeonhole principle states that at least 2 computers belong to 1 of the numbers 1..9.

i.e. $\exists$ at least 2 computers connected to the same # of computers.

3) (20 points) A coin is flipped ten times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly three heads?
c) contain at least three heads?
d) contain the same number of heads and tails?

a 2^{10}.

b $\binom{10}{3}$.

c $2^{10} - \left(\binom{10}{0} + \binom{10}{1} + \binom{10}{2}\right)$

d $\binom{10}{5}$

4) (10 points) Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.

$$\left(x+\frac{1}{x}\right)^{100} = \sum_{i=0}^{100} \binom{100}{i} x^{i} \left(\frac{1}{x}\right)^{100-i}$$

$$= \sum_{i=0}^{100} \binom{100}{i} x^{i} \cdot x^{i-100}$$

$$= \sum_{i=0}^{100} \binom{100}{i} x^{2i-100}$$

$$= \sum_{i=0}^{100} \binom{100}{i} x^{2(i-50)}$$

$\therefore$ the coefficient of x^k, a_k is as follows:

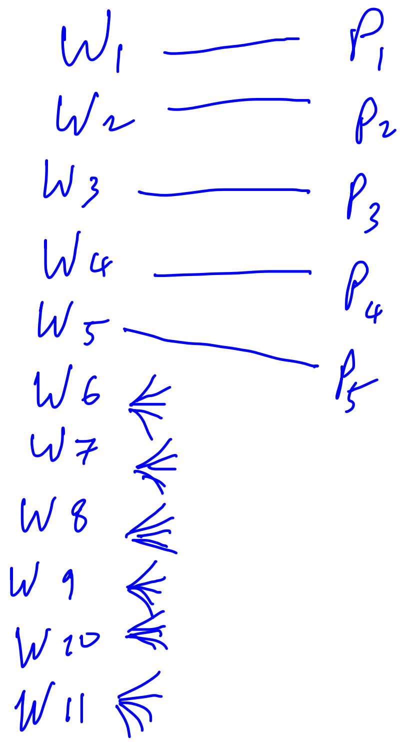
$$a_{2k+1} = 0 \quad \forall k \;\; -50 \le k \le 49$$

$$a_{2k} = \binom{100}{k+50} \;\; \forall k \;\; -50 \le k \le 50$$

since

$$2k = 2i - 100$$
$$2i = 2k + 100$$
$$i = k + 50$$

5) (10 points) Find the least number of cables required to connect eleven computers to five printers to guarantee that for every choice of five of the eleven computers, these five computers can directly access five different printers. Justify your answer.



$$\text{Total \# cables} = 5 + 6(5) = 35 \text{ cables.}$$

pf: Assume we have 34 cables. Then, there is a printer who is connected to only 4 computers (if not, each one is connected to at least 5 computers, for a total of $35 > 34$).

Hence, the remaining 4 printers will not be able to serve any 5 computers from the remaining 7 computers.

6) (20 points) How many strings of eight lowercase letters from the English alphabet contain
   a) the letter *a*?
   b) the letters *a* and *b*?
   c) the letters *a* and *b* in consecutive positions with *a* preceding *b*, with all the letters distinct?
   d) the letters *a* and *b*, where *a* is somewhere to the left of *b* in the string, with all the letters distinct?

(a) Total # of strings of 8 letters − # of strings without any a

$= 26^8 - 25^8$.

(b) Total # of strings of 8 letters − (# of strings without a + # of strings without b − # of strings without a ∧ without b)

$= 26^8 - (25^8 + 25^8 - 24^8)$

$= 26^8 + 24^8 - 2(25)^8$

(c) "Letters" are now c, d, ..., z (24) in addition to ab. Since there are 7 positions of ab, the total is $7 \cdot P(24, 6)$

(d) $\binom{8}{2}(24)(23)(22)(21)(20)(19)$

$= \binom{8}{2} P(24, 6)$.

7) (10 points) What is the probability that Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively, in a drawing if 80 people enter a contest and
   a) no one can win more than one prize.
   b) winning more than one prize is allowed.

(a) The sample space in this case is $P(80, 4)$.

$\therefore$ The probability $= \dfrac{1}{P(80,4)}$

(b) The sample space in this case is 80^4.

$\therefore$ The probability $= \dfrac{1}{80^4}$.

8) (25 points) What is the probability of these events when we randomly select a permutation of $\{1, 2, \dots, n\}$ where $n \geq 4$?

a) 1 precedes 2.
b) 2 precedes 1.
c) 1 immediately precedes 2.
d) n precedes 1 and $n-1$ precedes 2.
e) n precedes 1 and n precedes 2.

a) The # of different positions for 1 & 2 is $\binom{n}{2}$. Hence, the probability $= \dfrac{\binom{n}{2}(n-2)!}{n!}$

$$= \frac{n!}{2\,n!} = \frac{1}{2}.$$

b) Same probability as a) with the same exact reasoning.

c) $\dfrac{(n-1)!}{n!} = \dfrac{1}{n}.$

d) $p(n \text{ precedes } 1 \wedge n-1 \text{ precedes } 2) = p(n \text{ precedes } 1 \mid n-1 \text{ precedes } 2) \times p(n-1 \text{ precedes } 2).$

$n-1$ precedes 2 in $\binom{n}{2}(n-2)! = \dfrac{n!}{2}$.

or $p(n \text{ precedes } 1 \mid n-1 \text{ precedes } 2)$: Since we have $n-1$ & 2 are fixed with $n-1$ preceding 2. Then with $n-2$ positions left for n & 1, we have $\dfrac{(n-2)!}{2}$ permutations with n preceding 1. Now, counting the # of times $n-1$ precedes 2,

$$(n-1)\left\{\begin{array}{l} n-1 \;\; 2 \;\; \_ \;\; \_ \;\; \cdots \;\; \_ \\ n-1 \;\; \_ \;\; 2 \;\; \_ \;\; \cdots \;\; \_ \\ \vdots \\ n-1 \;\; \_ \;\; \cdots \;\; 2 \end{array}\right. \qquad \underbrace{\left\{\begin{array}{l} \_ \;\; n-1 \;\; 2 \;\; \_ \;\; \cdots \;\; \_ \\ \_ \;\; n-1 \;\; \_ \;\; 2 \;\; \cdots \;\; \_ \\ \vdots \\ \_ \;\; n-1 \;\; \_ \;\; \_ \;\; \cdots \;\; 2 \end{array}\right.}_{(n-2)} \quad \cdots \quad \cdots \quad 1$$

$$1+2+\cdots+n-1 = \frac{(n-1)n}{2}$$

$\therefore$ The # of times n precedes 1 given $n-1$ precedes 2 is

$$\frac{(n-1)n}{2}\cdot\frac{(n-2)!}{2} = \frac{n!}{4}.$$

$$\therefore\; p(n \text{ precedes } 1 \mid n-1 \text{ precedes } 2) = \frac{n!/4}{n!/2} = \frac{1}{2}$$

$$\therefore\; p(n \text{ precedes } 1 \wedge n-1 \text{ precedes } 2) = \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = \frac{1}{4}.$$

Another solution for (d): Since half of the permutations we have n precede 1, and half of the latter, we have $n-1$ precede 2, then probability of both happening is

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

(e) There are 3 cases for n, 1 & 2 in a permutation of $1 .. n$.

case 1: n precedes both 1 & 2.

case 2: n succeedes both 1 & 2.

case 3: n is in between 1 & 2.

Since there is no reason for one case to happen more than another, it is obvious that p (n preceding 1 & n preceding 2) $= \frac{1}{3}$.

(e) can also be solved in the way we did part (d) but it would be more complicated.