King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #3
(Due Friday July 27, 2018 at midnight)

I. Solve and submit the solutions to Questions 1-8. In all questions below, show your intermediate work. Make sure to attempt these questions without using a calculator.

1) (10 points) Prove or disprove that $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

Let $x = m + \epsilon_1$ & $y = n + \epsilon_2$ where $0 \le \epsilon_1 < 1$, $0 \le \epsilon_2 < 1$ & $m, n \in \mathbb{Z}$.

Then, $\lfloor x \rfloor = m$, $\lfloor y \rfloor = n$

$$\lfloor x+y \rfloor = \lfloor m + \epsilon_1 + n + \epsilon_2 \rfloor$$
$$= \lfloor m+n + \epsilon_1 + \epsilon_2 \rfloor$$
$$= m+n + \lfloor \epsilon_1 + \epsilon_2 \rfloor$$

We have 2 cases: if $\epsilon_1 + \epsilon_2 < 1$

$\lfloor x+y \rfloor = m+n$

otherwise if $1 \le \epsilon_1 + \epsilon_2 < 2$ (can never be equal to 2)

then $\lfloor x+y \rfloor = m+n+1$.

$\lfloor 2x \rfloor = \lfloor 2m + 2\epsilon_1 \rfloor = 2m + \lfloor 2\epsilon_1 \rfloor$

$\lfloor 2y \rfloor = \lfloor 2n + 2\epsilon_2 \rfloor = 2n + \lfloor 2\epsilon_2 \rfloor$

Again we have 2 cases: if $\epsilon_1 + \epsilon_2 < 1$ then at least one of ϵ_1 or ϵ_2 is less than 0.5 & hence $\lfloor 2x \rfloor + \lfloor 2y \rfloor \le 2m + 2n + 1$ & hence the inequality holds.

otherwise if $\epsilon_1 + \epsilon_2 > 1$, then at least one of ϵ_1 or $\epsilon_2 > 0.5$. i.e. $\lfloor 2x \rfloor + \lfloor 2y \rfloor \ge 2m + 2n + 1$

i.e. $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$

2) (15 points) Give an example of two uncountable sets A and B such that $A - B$ is

a) finite.

$[0,1] - [0,1) = \{1\}$

b) countably infinite.

$[0,1] \cup \mathbb{Z} - (0,1) = \mathbb{Z}$

c) uncountable.

$[0,1] - [0,\frac{1}{2}] = (\frac{1}{2},1]$

3) (10 points) Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

Assume that B is countable. Then there is a 1:1 correspondance from $B \to \mathbb{N}$ i.e. elements of B can be listed as $e_1, e_2, \ldots$

Since $A \subseteq B$, A can be represented as a subsequence of $e_1, e_2, \ldots$, which is a contradiction to the fact that A is uncountable.

4) (15 points) Use mathematical induction to show that $\frac{1}{2} \cdot \frac{3}{4} \dots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$ for all positive integers $n \geq 2$.

Basis: $P(2)$: $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} < \frac{1}{\sqrt{7}}$ which is true.

Induction step: Assume that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$ $\forall\, n \geq 2$.

to show that $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n+4}}$.

by the induction hypothesis,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} \cdot \frac{2n+1}{2n+2} < \frac{1}{\sqrt{3n+1}} \cdot \frac{2n+1}{2n+2}$$

$$= \frac{2n+1}{\sqrt{(3n+1)(2n+2)^2}}$$

$$= \frac{2n+1}{\sqrt{(3n+1)(4n^2+8n+4)}}$$

$$= \frac{2n+1}{\sqrt{12n^3+28n^2+20n+4}}$$

$$= \frac{2n+1}{\sqrt{(3n+4)(2n+1)^2+n}}$$

$$< \frac{2n+1}{\sqrt{(3n+4)(2n+1)^2}}$$

$$< \frac{1}{\sqrt{3n+4}}$$

$$\begin{array}{r} 3n+4 \\ 4n^2+4n+1 \overline{)\,12n^3+28n^2+20n+4} \\ -12n^3 \mp 12n^2 \mp 3n \\ \hline 16n^2+17n+4 \\ -16n^2 \mp 16n \mp 4 \\ \hline n \end{array}$$

5) (10 points) Which amounts of money can be formed using just two-riyal bills and seven-riyal bills? Prove your answer using strong induction.

All money integer values ≥ 9.

pf:

Basis: $P(9)$: $1(2) + 1(7) = 9$

$P(10)$: $5(2) + 0(7) = 10$

Induction step. Assume that $\exists\, i, j \in \mathbb{N}$ s.t.

$$2i + 7j = k \quad 10 \leq k \leq n$$

To show that $\exists\, i', j' \in \mathbb{N}$ s.t.

$$2i' + 7j' = n+1.$$

Using induction hypothesis, $\exists\, i, j \in \mathbb{N}$ s.t. $2i + 7j = n-1$.

adding 1 2-riyal to both sides yields

$$2(i+1) + 7j = n+1$$

$\therefore$ $i' = i+1$.

$j' = j$.

6) (20 points) Give a recursive definition of the sequence $\{a_n\}, n = 1, 2, 3, \ldots$ if

a) $a_n = 6n + 7$

$a_1 = 13.$

$a_{n+1} = 6(n+1) + 7 = 6n + 13 = 6n + 7 + 6$
$= a_n + 6 \quad n \geq 1.$

$$\therefore \quad a_n = \begin{cases} 13 & n = 1 \\ a_{n-1} + 6 & n > 1 \end{cases}$$

b) $a_n = n^2 - 2$

$a_1 = -1$

$a_{n+1} = (n+1)^2 - 2 = n^2 + 2n + 1 - 2 = (n^2 - 2) + 2n + 1 = a_n + 2n + 1$
$n \geq 1$

$$\therefore \quad a_n = \begin{cases} -1 & n = 1 \\ a_{n-1} + 2n + 1 & n > 1 \end{cases}$$

7) (10 points) Give a recursive definition of the set of positive integers not divisible by 4.

Basis: $1 \in S, 2 \in S, 3 \in S.$

Recursive definition: $\forall x \in S, \quad x + 4 \in S$

8) (10 points) Let f_n be the n^{th} Fibonacci number. If n is a positive integer, prove that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

Using Mathematical Induction.

Basis: $f_1^2 = 1^2 = f_1 f_2 = (1)(1)$ holds.

Inductive step: Assume that the result holds $\forall\, n \geq 1$.

To show that

$$f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 = f_{n+1} f_{n+2}.$$

$$\left(f_1^2 + f_2^2 + \cdots + f_n^2\right) + f_{n+1}^2 = f_n f_{n+1} + f_{n+1}^2 \quad (\text{Induction hypothesis})$$

$$= f_{n+1}\left(f_n + f_{n+1}\right)$$

$$= f_{n+1} f_{n+2}$$

$$(f_n + f_{n+1} = f_{n+2}).$$