King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #3
(Due Wednesday June 26, 2013 at midnight)

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1) (6 points) Use rules of inference to show that if

$$\forall x(P(x) \vee Q(x)) \text{ and } \forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$$

are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

The above is equivalent to $\forall x(P(x) \vee Q(x))$ and $\forall x((P(x) \vee \neg Q(x)) \vee R(x))$ which is equivalent to $\forall x((P(x) \vee R(x)) \vee \neg Q(x))$ by the commutative and associative properties. Now, using resolution, $\forall x(P(x) \vee Q(x))$ and $\forall x((P(x) \vee R(x)) \vee \neg Q(x))$ imply that $\forall x(P(x) \vee (P(x) \vee R(x)))$ is true, which is equivalent to $\forall x(P(x) \vee R(x))$ which is equivalent to $\forall x(\neg R(x) \rightarrow P(x))$.

(0, 3, 6)

2) (4 points) Prove that the statement that every positive integer can be written as the sum of the squares of three integers is not correct.

7 is a positive integer. Since the smallest squares are 0,1,4 and 9, and since squares are non-negative, $7 = x^2 + y^2 + z^2$ iff $\exists x,y,z \in \{0, 1, 2\}$ s.t. $7 = x^2 + y^2 + z^2$.
Now, since 22 + 22 = 8, we can have at most one "2" among them. In addition they cannot be all zeros or ones, as the maximum sum in that case is equal to 3. Now, looking at sums of three squares, with one 2, we get sums of 4, 5, 6. Any greater sum value must include another 2, which will render the sum to be equal to 8 and above. Hence 7 cannot be represented as the sum of 3 squares.

(0, 2, 4)

3) (7 points) Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$ **T**
b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$ **T**
c) $\{\emptyset\} \in \{\emptyset\}$ **F**
d) $\{\emptyset\} \in \{\{\emptyset\}\}$ **T**
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **T**
f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$ **T**
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$ **F**

(1 point each)

4) (3 points) Find two sets A and B such that $A \in B$ and $A \subseteq B$.

$A = \{\emptyset\}$, and $B = \{\emptyset, \{\emptyset\}\}$

(0, 3)

5) (9 points) Let A, B, and C be sets. Show that (not using membership tables or Venn diagrams)

   a) (3 points) $(A - B) - C \subseteq A - C$.

   b) (6 points) $(B - A) \cup (C - A) = (B \cup C) - A$.

a) $x \in (A-B) - C \Leftrightarrow x \in (A \cap \bar{B}) \cap C$

$\Leftrightarrow x \in (A \wedge \bar{C}) \cap \bar{B}$ Commutative + Associative

(0,3)

$\rightarrow x \in A \cap \bar{C}$ Simplification.

$\Leftrightarrow x \in A - C.$

b) $x \in (B - A) \cup (C - A) \Leftrightarrow x \in B - A \vee x \in C - A$

$\Leftrightarrow x \in (B \cap \bar{A}) \vee x \in (C \wedge \bar{A})$

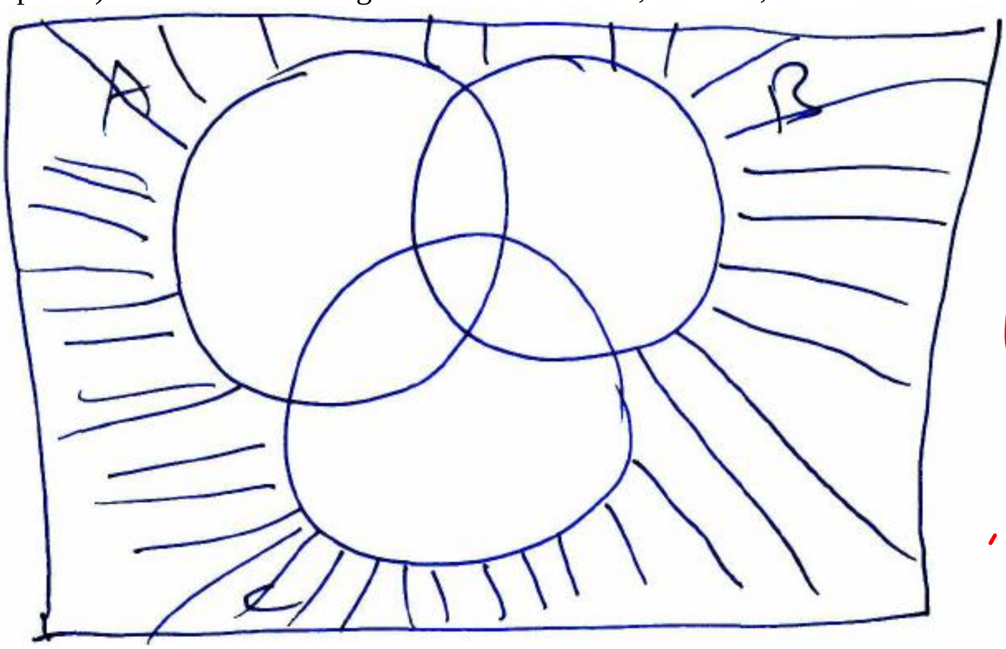
(0,3,6)

$\Leftrightarrow x \in (B \wedge \bar{A}) \cup (C \wedge \bar{A})$

$\Leftrightarrow x \in (B \cup C) \cap \bar{A}$

$\Leftrightarrow x \in (B \cup C) - A.$

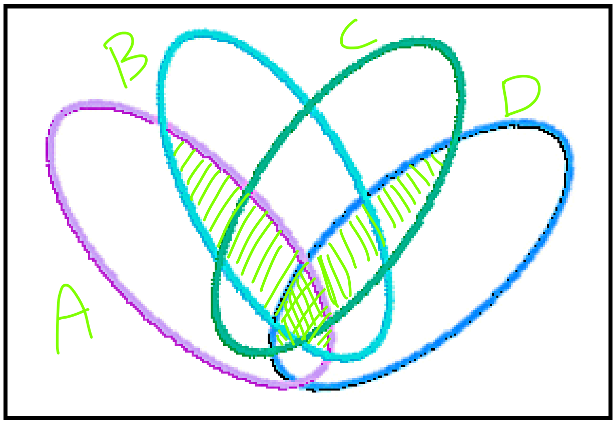
6) (4 points) Draw the Venn diagram for $\bar{A} \cap \bar{B} \cap \bar{C}$, where A, B and C are sets.

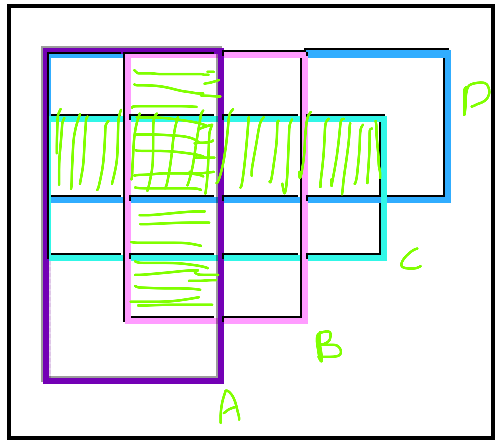


(0,4)

Count the number of subsets in the Venn diagram, say n. then, give ceiling[(n/16)*7] points.

7) (6 points) Draw the Venn diagram for (A ∩ B) ∪ (C ∩ D) where *A*, *B*, *C* and *D* are sets.





8) (6 points) Show that A ⊕ B = (A − B) ∪ (B − A).

$x \in A \oplus B \leftrightarrow x \in (A \cup B) - (A \cap B)$

$\leftrightarrow x \in (A \cup B) \cap \overline{(A \cap B)}$

$\leftrightarrow x \in (A \cup B) \cap (\overline{A} \cup \overline{B})$

$\leftrightarrow x \in (A \cup B) \land x \in (\overline{A} \cup \overline{B})$

$\leftrightarrow [(x \in A) \lor (x \in B)] \land [(x \in \overline{A}) \lor (x \in \overline{B})]$

$\leftrightarrow [[x \in A \lor x \in B] \land (x \in \overline{A})] \lor ((x \in A \lor x \in B) \land (x \in \overline{B})]$

$\leftrightarrow [(x \in A \land x \in \overline{A}) \lor (x \in B \land x \in \overline{A})] \lor [(x \in A \land x \in \overline{B}) \lor (x \in B \land x \in \overline{B})]$

$\leftrightarrow [x \in \Phi \lor (x \in B \land x \in \overline{A})] \lor [(x \in A \land x \in \overline{B}) \lor (x \in \Phi)$

$\leftrightarrow (x \in B \land x \in \overline{A}) \lor (x \in A \land x \in \overline{B})$

$\leftrightarrow (x \in B \cap \overline{A}) \lor (x \in A \cap \overline{B})$

$\leftrightarrow (x \in B - A) \lor (x \in A - B)$

$\leftrightarrow x \in (B - A) \cup (A - B)$

(0,3,6)

9) (12 points) Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) (3 points) $A_i = \{i, i + 1, i + 2, \ldots\}$.

$\bigcup_{i=1}^{\infty} A_i = A_1 = \{1,2,3,\ldots\}$ — (0,1)

$\bigcap_{i=1}^{\infty} A_i = \Phi$ — (0,2)

b) (3 points) $A_i = \{0, i\}$.

$\bigcup_{i=1}^{\infty} A_i = \{0,1,2,3,\ldots\}$ — (0,1)

$\bigcap_{i=1}^{\infty} A_i = \{0\}$ — (0,2)

c) (3 points) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.

$\bigcup_{i=1}^{\infty} A_i = R^+$. — (0,1)

$\bigcap_{i=1}^{\infty} A_i = A_1 = (0,1)$ — (0,2)

d) (3 points) $A_i = (i,\infty)$, that is, the set of real numbers x with $x > i$.

$\bigcup_{i=1}^{\infty} A_i = (1,\infty)$ — (0,1)

$\bigcap_{i=1}^{\infty} A_i = \Phi$ — (0,2)

10) (5 points) Find the domain and range of the function that assigns to each positive integer the largest integer not exceeding the square root of the integer.

$1 \to 1$
$2 \to 1$
$3 \to 1$
$4 \to 2$
$5 \to 2$
$6 \to 2$
$7 \to 2$
$8 \to 2$
$9 \to 3$
$\vdots$

Domain: Z^+ — (0,2)

Range: Z^+ — (0,3)

11) (4 points) Find these values

a) (2 points) $\left\lfloor \frac{1}{2} + \left\lceil \frac{1}{2} \right\rceil \right\rfloor$

1 (0, 2)

b) (2 points) $\left\lfloor \left\lfloor -\frac{1}{2} \right\rfloor + \left\lceil \frac{1}{2} \right\rceil - 1 \right\rfloor$

-1 (0, 2)

12) (12 points) Determine whether each of these functions is a bijection from **R** to **R**. Clearly justify your answer.

a) (6 points) $f(x) = -2x + 8$

1. $f(a) = f(b) \Leftrightarrow -2a+8 = -2b+8$
$\Leftrightarrow -\cancel{2}a = -\cancel{2}b$
$\Leftrightarrow a = b \rightarrow f$ is 1:1

(0, 3, 6)

2. Let $y \in \mathbb{R}$
$y = -2x+8$
$y - 8 = -2x$
$\frac{8-y}{2} = x$ where $x \in \mathbb{R}$

Since $f\left(\frac{8-y}{2}\right) = y$, f is onto
$\therefore f$ is a 1:1 correspondance.

b) (6 points) $f(x) = \frac{x}{x^2+1}$

$f(a) = f(b) \Leftrightarrow \frac{a}{a^2+1} = \frac{b}{b^2+1} \Leftrightarrow a(b^2+1) = b(a^2+1)$
$\Leftrightarrow ab^2 - ba^2 + a - b = 0$
$\Leftrightarrow ab(b-a) + (a-b) = 0$
$\Leftrightarrow (b-a)(ab-1) = 0$
$\Leftrightarrow b-a=0 \rightarrow a=b$
or $ab-1=0 \rightarrow a = \frac{1}{b}$

(0, 3, 6)

$\therefore f(2) = \frac{2}{4+1} = \frac{2}{5}$

$f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2+1} = \frac{\frac{1}{2}}{\frac{1+4}{4}} = \frac{2}{5}$

since $f\left(\frac{1}{2}\right) = f(2)$, f is not 1:1

$\therefore f$ is not a 1:1 correspond.

13) (6 points) Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

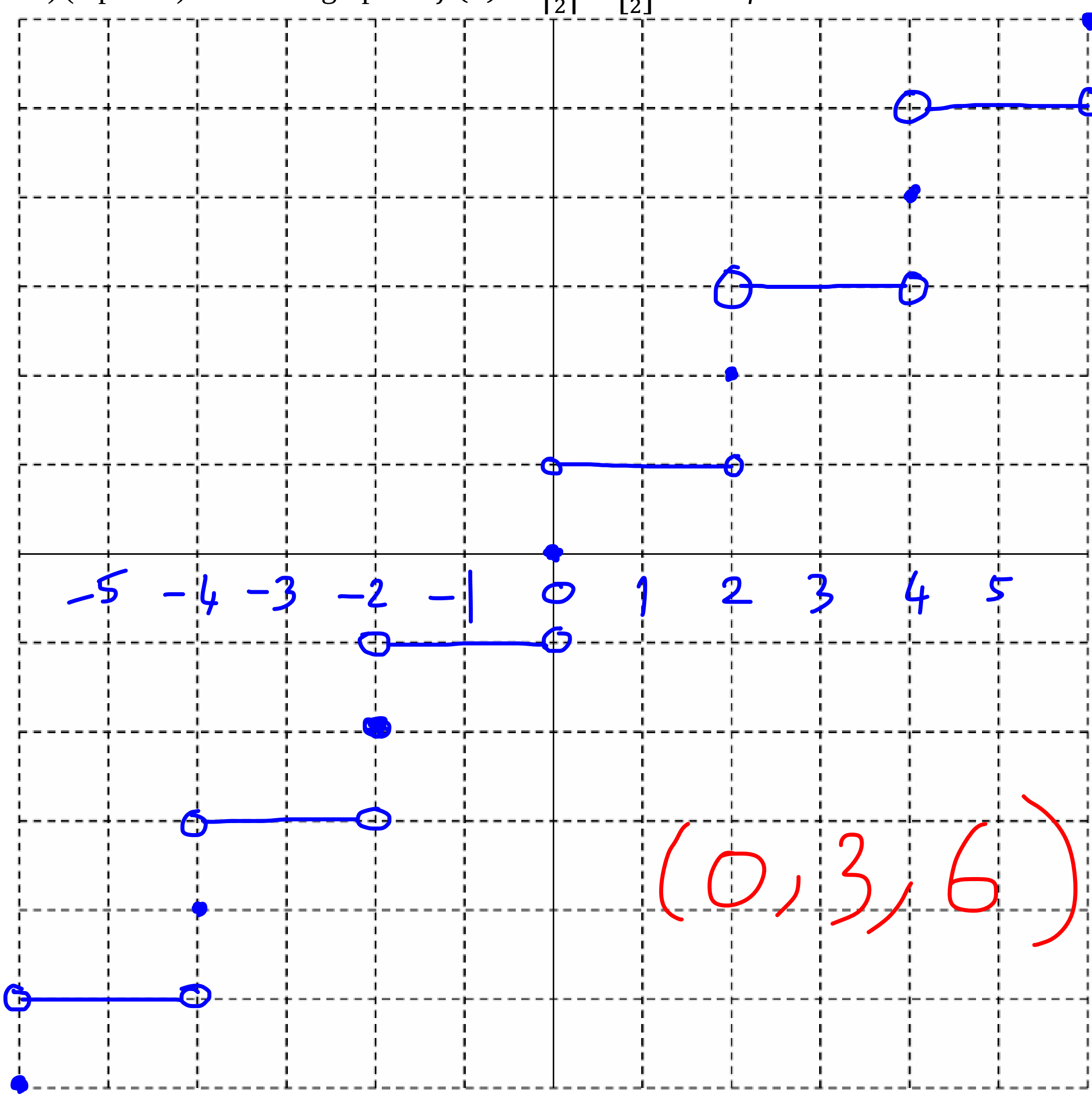
$f \circ g = f(cx+d) = a(cx+d)+b = acx+ad+b$

$g \circ f = g(ax+b) = c(ax+b)+d = acx+cb+d$

$f \circ g = g \circ f \Leftrightarrow ad+b = cb+d$

(0, 3, 6)

14) (6 points) Draw the graph of $f(x) = \left\lceil \frac{x}{2} \right\rceil + \left\lfloor \frac{x}{2} \right\rfloor$ where f is from **R** to **R**.



$\left\lceil \frac{x}{2} \right\rceil = 1 \Leftrightarrow 0 < \frac{x}{2} \le 1$

$\Leftrightarrow 0 < x \le 2$

$\left\lfloor \frac{x}{2} \right\rfloor = 0 \Leftrightarrow 0 \le \frac{x}{2} < 1$

$\Leftrightarrow 0 \le x < 2$

$\left\lceil \frac{x}{2} \right\rceil = 2 \Leftrightarrow 1 < \frac{x}{2} \le 2$

$\Leftrightarrow 2 < x \le 4$

$\left\lfloor \frac{x}{2} \right\rfloor = 1 \Leftrightarrow 1 \le \frac{x}{2} < 2$

$\Leftrightarrow 2 \le x < 4$

15) (6 points) Prove or disprove that $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ for all real numbers x.

$\left\lceil \frac{x}{2} \right\rceil = n \iff n-1 < \frac{x}{2} \le n \iff 2n-2 < x \le 2n$

$\left\lfloor \frac{x+1}{2} \right\rfloor = n \iff n \le \frac{x+1}{2} < n+1 \iff 2n \le x+1 < 2n+2 \iff 2n-1 \le x <$

$\iff 2n-1 \le x < 2n+1$

(0/6)

$2n-2 \quad 2n-1 \quad 2n \quad 2n+1$

Let $x = 6.5$

Then $\left\lceil \frac{6.5}{2} \right\rceil = 4$ & $\left\lfloor \frac{6.5+1}{2} \right\rfloor = 3$

$3 \neq 4 \implies$ They are not equal

16) (4 points) Find a recurrence relation satisfied by the sequence $a_n = n^2 + n$

$a_n = n^2+n$, $a_{n-1} = (n-1)^2+(n-1) = n^2-2n+1+n-1 - n^2 - n$

$\therefore a_n = a_{n-1} + 2n$

(0/4)

OR

$a_{n+1} = (n+1)^2+(n+1) = n^2+2n+1+n+1 = n^2+3n+2 = a_n + 2(n+1)$

$\therefore a_{n+1} = a_n + 2(n+1)$

In both cases, $a_1 = 2$.