King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #4
(Due Friday July 5, 2013 at midnight)

Answer the following questions. NOTE: There are 10 extra points in this assignment.

1. (6 points) What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals
   a. $(-2)^n$?
   b. 3?
   c. $2^n + \left\lceil \frac{3n}{2} \right\rceil$?

a. $a_0 = 1,\ a_1 = -2,\ a_2 = 4,\ a_3 = -8$. 2 pts

b. $a_0 = a_1 = a_2 = a_3 = 3$ 1 pt

c. $a_0 = 1,\ a_1 = 2 + \left\lceil \frac{3}{2} \right\rceil = 4$ 1 pt

$a_2 = 4 + \left\lceil \frac{6}{2} \right\rceil = 7$ 1 pt

$a_3 = 8 + \left\lceil \frac{9}{2} \right\rceil = 8 + 5 = 13$ 1 pt

2. (6 points) Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

2 pts 1. $a_n = 2n + 3,\ n = 0, 1, 2, \ldots$ $a_3 = 9$

2 pts 2. $a_n = a_{n-1} + \left\lfloor \frac{n+3}{2} \right\rfloor$, $a_0 = 3$ $\Rightarrow$ $a_1 = 3 + 2 = 5$, $a_2 = 5 + \left\lfloor \frac{5}{2} \right\rfloor = 7$, $a_3 = 7 + \left\lfloor \frac{6}{2} \right\rfloor = 10$

2 pts 3. The n^{th} term is the n^{th} prime, where n starts from 2:

3, 5, 7, 11, 13, ... etc.

(or any other correct solution)

(12 points) Assume that the population of the world in 2010 was 6.9 billion and is growing at the rate of 1.1% a year.

a. (4 points) Set up a recurrence relation for the population of the world *n* years after 2010.
b. (6 points) Find an explicit formula for the population of the world *n* years after 2010.
c. (2 points) What will the population of the world be in 2030?

a. $G(n) = G(n-1) + 0.011\,G(n-1) = 1.011\,G(n-1)$ (0, 3)

$G(0) = 6900000000$ (0, 1)

b. $G(n) = 1.011\,G(n-1)$

$= 1.011(1.011\,G(n-2))$

$= (1.011)^2 G(n-2)$ (0, 6)

$\vdots$

$= (1.011)^n G(0)$ (depends on a)

c. $G(20) = (1.011)^{20}(6900000000)$ (0, 2)

$= 8{,}587{,}607{,}815$ depends on b

3. (9 points) For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next term of the sequence.
   a. 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, . . .

$a_n = 3^n - 1$ $\qquad a_{10} = 59048$

2 pts $\qquad$ 1 pt

   b. 2, 4, 16, 256, 65536, 4294967296, . . .

$a_n = 2^{2^n}$ $\qquad a_6 = 1.8446744073709551616 \times 10^{19}$

2 pts $\qquad$ 1 pt

   c. 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, . . .

$a_0 = 1,\ a_n = (2n+1)a_{n-1}$ $\qquad a_9 = 654{,}729{,}075$

2 pts $\qquad$ 1 pt

4. (6 points) Find the value of the double sum, $\sum_{i=12}^{40} \sum_{j=17}^{30} 2i^2 j^3$ showing all your work.

$$= 2\sum_{i=12}^{40} i^2 \left(\sum_{j=17}^{30} j^3\right) = 2\sum_{i=12}^{40} i^2 \left[\sum_{j=1}^{30} j^3 - \sum_{j=1}^{16} j^3\right]$$

$$= 2\sum_{i=12}^{40} i^2 \left[\left[\frac{(30)(31)}{2}\right]^2 - \left[\frac{(16)(17)}{2}\right]^2\right]$$

$$= 2\sum_{i=12}^{40} i^2 (197729)$$

$$= 2(197729)\sum_{i=12}^{40} i^2$$

$$= 2(197729)\left(\sum_{i=1}^{40} i^2 - \sum_{i=1}^{11} i^2\right)$$

$$= 2(197729)\left(\frac{40(41)(81)}{6} - \frac{11(12)(23)}{6}\right)$$

$$= 2(197729)(21634) = 8555338372$$

(0, 3, 6)

5. (14 points)Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.
    a. the integers greater than 10
    b. the odd negative integers
    c. the integers with absolute value less than 1,000,000
    d. the real numbers between 0 and 2
    e. the set $A \times \mathbf{Z}^+$ where $A = \{2, 3\}$
    f. the integers that are multiples of 10

a. Countably infinite. $f(n) = n+9$ 2
1 (or $f(n) = n-9$)

b. Countably Infinite. 1 $f(n) = -2n+1$
2

c. Finite. 1

d. Uncountable 1

e. Countably infinite 1
$f(n) = (2, \frac{n}{2})$ where n is even
$= (3, \frac{n+1}{2})$ where n is odd
2

f. Countably infinite : $f(n) = 10n$.
1 2

6. (6 points) Give an example of two uncountable sets A and B such that $A - B$ is
   a. finite.
   b. countably infinite.
   c. uncountable.

(0, 2)

a. $A = [1,2]$
$B = (1,2)$

b. $A = \mathbb{R}$.
$B = \bar{\mathbb{Q}} = \mathbb{R} - \mathbb{Q}$ [irrationals]

(0, 2)

c. $A = (1,3)$
$B = (1,2)$.

(0, 2)

or any correct answer.

7. (6 points) Prove that $\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\,.\,2^n}$ whenever n is a nonnegative integer.

Using Mathematical induction

1. Basis: $n=0$

$$\sum_{j=0}^{0}\left(-\frac{1}{2}\right)^j = \left(-\frac{1}{2}\right)^0 = 1 \stackrel{?}{=} \frac{2^1+1}{3\,.\,1} = \frac{3}{3} = 1 \checkmark$$

(0, 2)

2. Inductive step: Assume that $P(n)$ holds, i.e,

$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\,.\,2^n}$. To show that $\sum_{j=0}^{n+1}\left(-\frac{1}{2}\right)^j = \frac{2^{n+2}+(-1)^{n+1}}{3\,.\,2^{n+1}}$.

$$\sum_{j=0}^{n+1}\left(-\frac{1}{2}\right)^j = \sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j + \left(-\frac{1}{2}\right)^{n+1} = \frac{2^{n+1}+(-1)^n}{3\,.\,2^n} + (-1)^{n+1}\left(\frac{1}{2}\right)^{n+1} \text{ (By I.H.)}$$

$$= \frac{2^{n+1}+(-1)^n}{3\,.\,2^n} + \frac{(-1)^{n+1}}{2^{n+1}} = \frac{2^{n+2}+2(-1)^n+3(-1)^{n+1}}{3\,.\,2^{n+1}} = \frac{2^{n+2}+(-1)^n[2-3]}{3\,.\,2^{n+1}}$$

$$= \frac{2^{n+2}+(-1)^{n+1}}{3\,.\,2^{n+1}}$$

(0, 2, 4)

8. (6 points) Prove that $\frac{1}{2n} \le \frac{1.3.5.....(2n-1)}{2.4.6.....2n}$ whenever n is a positive integer.

By Mathematical induction (0, 2)

Basis: $n=1$. $\frac{1}{2} \overset{?}{\le} \frac{1}{2}$ which is true.

Induction Hypothesis: Assume that $\frac{1}{2n} \le \frac{1.3.5\ldots(2n-1)}{2.4.6,\ldots 2n}$ holds. To show that $\frac{1}{2(n+1)} \le \frac{1.3.5\ldots(2n+1)}{2.4.6.\ldots(2n+2)}$.

$$\frac{1}{2(n+1)} = \frac{1}{2n+2} = \frac{1}{2n} + x \Leftrightarrow x = \frac{1}{2n+2} - \frac{1}{2n} = \frac{n-(n+1)}{2n(n+1)} = \frac{-1}{2n(n+1)}$$

$$\therefore \frac{1}{2(n+1)} = \frac{1}{2n} - \frac{1}{2n(n+1)} = \frac{1}{2n}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2n}\left(\frac{n}{n+1}\right)$$

$$\le \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} \cdot \frac{n}{n+1} = \frac{1.3\ldots(2n-1).\,2n}{2.4\ldots 2n \cdot 2(n+1)} \quad \text{By I.H.}$$

$$\le \frac{1.3\ldots(2n-1)(2n+1)}{2.4\ldots 2n \cdot 2(n+1)} \quad \text{since } 2n \le 2n+1.$$

(0, 2, 4)

9. (10 points) Let *P(n)* be the statement that a postage of *n* cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that *P(n)* is true for $n \geq 18$.

   a. (2 points) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.
   b. (2 points) What is the inductive hypothesis of the proof?
   c. (2 points) What do you need to prove in the inductive step?
   d. (4 points) Complete the inductive step for $k \geq 21$.

a. $P(18): 1(4)+2(7)$ $\quad$ $P(19): 3(4)+1(7)$ $\quad$ 0.5 each

$P(20): 5(4)+0(7)$ $\quad$ $P(21): 0(4)+3(7)$

b. $\forall j, 18 \leq j \leq n$, a postage of j cents can be formed using 4-cent & 7-cent stamps $\quad$ (0, 2)

c. To show that $P(n+1)$ holds, i.e. a postage of $n+1$ can be formed using 4-cent & 7-cent stamps. $\quad$ (0, 2)

d. Since $P(j)$ holds $\forall j$ $18 \leq j \leq n$, choose $P(n-3)$. Using I.H. We can find 4-cent & 7-cent stamps to form $n-3$ cents. Adding one 4-cent stamp will result in $n+1$ cents in stamps.

(0, 2, 4)

10. (4 points) Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$
if $a_n = n(n + 1)$.

$a_n = n^2+n$, $a_{n-1} = (n-1)n = n^2-n = n^2+n-2n = a_n - 2n$

$$\therefore a_n = \begin{cases} a_{n-1} + 2n & n>1 \\ 2 & n=1 \end{cases}$$

(0/4)

11. (4 points) Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if $a_n = n^2$.

$a_n = n^2$, $a_{n-1} = (n-1)^2 = n^2-2n+1 = a_n - 2n+1$

$$\therefore a_n = \begin{cases} a_{n-1} + 2n - 1 & n>1 \\ 1 & n=1 \end{cases}$$

(0/4)

12. (6 points) Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_{2n}$ when n is a positive integer.

There is a typo in this question. $f_1^2+f_2^2+\cdots+f_n^2 = f_n f_{n+1}$ (not f_{2n}).

Basis: $n=1$: $f_1^2 = 1 \stackrel{?}{=} f_1 f_2 = 1(1) = 1$ ✓

Induction: Assume that $f_1^2+f_2^2+\cdots+f_n^2 = f_n f_{n+1}$. To show that $f_1^2+f_2^2+\cdots+f_{n+1}^2 = f_{n+1}f_{n+2}$

$f_1^2+\cdots+f_{n+1}^2 = f_1^2+f_2^2+\cdots+f_n^2+f_{n+1}^2$

$= f_n f_{n+1} + f_{n+1}^2$ (by Induction hypothesis)

$= f_{n+1}(f_n + f_{n+1}) = f_{n+1} f_{n+2}$ (by Fibonacci definition)

All students get +6.

13. (15 points) Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a, b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$.

a. (3 points) List the elements of S produced by the first four applications of the recursive definition.

b. (6 points) Use strong induction on the number of applications of the recursive step of the definition to show that $a \leq 2b$ whenever $(a, b) \in S$.

c. (6 points) Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

2 points if only 4 elements are listed.

a. (0,1), (1,1), (2,1)

(0,2), (1,2), (2,2), (3,2), (4,2)

(0,3), (1,3), (2,3), (3,3), (4,3), (5,3), (6,3)

(0,4), (1,4), (2,4), (3,4), (4,4), (5,4), (6,4), (7,4), (8,4)

b. Let $P(n)$ be the statement that $a \leq 2b$ whenever $(a,b) \in S$ is obtained after n applications of the rules.

Basis: $P(0)$ is true as $0 \leq 2(0) = 0$ (0/2)

Inductive step: Assume $P(j)$ holds for $1 \leq j \leq n$. To show that $P(n+1)$ holds.

1 point Let $(t,u) \in S$ be obtained by $n+1$ applications of the recursive step.

Then, (t,u) has been obtained from $(a,b) \in S$ by one of the following rules:

1. $(t,u) = (a, b+1)$

1 pt By I.H.: $a \leq 2b \Rightarrow a \leq 2b+2 = 2(b+1)$, i.e. $t \leq 2u$

2. $(t,u) = (a+1, b+1)$

1 pt By I.H.: $a \leq 2b \Leftrightarrow a+1 \leq 2b+1 \Rightarrow a+1 \leq 2b+2 = 2(b+1)$, i.e. $t \leq 2u$

3. $(t,u) = (a+2, b+1)$

1 pt By I.H.: $a \leq 2b \Leftrightarrow a+2 \leq 2b+2 = 2(b+1)$, i.e. $t \leq 2u$.

c. By Structural Induction:

1. Basis: For $(0,0)$, $0 \leq 2(0) = 0$ ✓ 2 pts

2. Inductive step:

Assume it holds for $(a,b) \in S$ (i.e. $a \leq 2b$), then 1 pt

1 pt $a \leq 2b \Rightarrow a \leq 2b + 2 = 2(b+1)$. i.e. $(a, b+1) \in S$

1 pt $a \leq 2b \Leftrightarrow a+1 \leq 2b+1 \leq 2b+2$. i.e. $(a+1, b+1) \in S$

1 pt $a \leq 2b \Leftrightarrow a+2 \leq 2b+2$, i.e. $(a+2, b+1) \in S$.