King Fahd University of Petroleum and Minerals
Information and Computer Science Department

ICS 253: Discrete Structures I

Homework Assignment #6
(Due Monday July 22, 2013 at midnight)

Answer the following questions. NOTE: There are 10 extra points in this assignment.

1. (40 points) Section 6.3:
   a. (5 points) In how many ways can a set of two positive integers less than 100 be chosen?

      $\binom{99}{2}$ (2,5)

   b. (5 points) How many subsets with an odd number of elements does a set with 10 elements have?

      $\binom{10}{1}+\binom{10}{3}+\binom{10}{5}+\binom{10}{7}+\binom{10}{9}$ (2,5)

   c. (12 points) A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
      i. (3 points) are there in total?

         $2^8$ (0,3)

      ii. (3 points) contain exactly three heads?

         $\binom{8}{3}$ (0,3)

      iii. (3 points) contain at least three heads?

         $2^8-\binom{8}{0}-\binom{8}{1}-\binom{8}{2}$ (0,3)

      iv. (3 points) contain the same number of heads and tails?

         $\binom{8}{4}$ (0,3)

d. (18 points) How many permutations of the letters ABCDEFGH contain

i. (3 points) the string ED?

$7!$ (0,3)

ii. (3 points) the string CDE?

$6!$ (0,3)

iii. (3 points) the strings BA and FGH?

$5!$ (0,3)

iv. (3 points) the strings AB, DE, and GH?

$5!$

v. (3 points) the strings CAB and BED?

(1,3)

Since CAB & BED are both substrings, it follows that the substring should by CABED (The only way both can be substrings). $\therefore$ $4!$

vi. (3 points) the strings BCA and ABF?

(0,3)

$0$

2. (20 points) Section 6.4:

a. (5 points) Find the expansion of $(x + y)^5$

(0,5)

$\binom{5}{0}x^5+\binom{5}{1}x^4y+\binom{5}{2}x^3y^2+\binom{5}{3}x^2y^3+\binom{5}{4}xy^4+\binom{5}{0}y^5$

$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

b. (5 points) Find the coefficient of $x^5y^8$ in $(2x - y)^{13}$.

(0,5)

$\binom{13}{5}(2x)^5(-y)^8=\binom{13}{5}2^5x^5y^8$

$\therefore$ Coefficient $=\binom{13}{5}2^5$

c. (10 points) Give a formula for the coefficient of $x^k$ in the expansion of $(x + 1/x)^{100}$, where $k$ is an integer.

$\binom{100}{j} x^{100-j} \left(\frac{1}{x}\right)^j = \binom{100}{j} x^{100-2j}$ (0,3)

since $0 \le j \le 100$, then $100 \geq 100-2j \geq -100$

Let $k = 100-2j \Leftrightarrow 2j = 100-k \leftrightarrow j = \frac{100-k}{2}$

$\therefore$ Coefficient of $x^k$ is $\binom{100}{\frac{100-k}{2}}$ (2,7)

Note that $k$ runs from $-100$ to $100$ in increments of 2 (rest have 0 as the coefficient)

3. (18 points) Section 7.1:

a. (6 points) What is the probability that a five-card poker hand contains the ace of hearts?

$(1)\binom{51}{4}/\binom{52}{5}$ (2,6)

b. (6 points) What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds (i.e. <1-2-3-4-5, 2-3-4-5-6, …, 9-10-J-Q-K, 10-J-Q-K-1>?

$\binom{4}{1}(10)/\binom{52}{5}$ (2,6)

c. (6 points) In a superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer.

$\binom{11}{7}/\binom{80}{7}$

which is also equal to

$\binom{73}{4}/\binom{80}{11}$ (2,6)

4. (32 points) Section 7.2:
   a. (9 points) What is the probability of these events when we randomly select a permutation of {1, 2, 3}?
      i. (3 points) 1 precedes 3.
      ii. (3 points) 3 precedes 1.
      iii. (3 points) 3 precedes 1 and 3 precedes 2.

Permutations: 123, 132, 213, 231, 312, 321

(i) 3/6 (0,3)

(ii) 3/6 (0,3)

(iii) 2/6 (0,3)

   b. (6 points) Suppose that $E$ and $F$ are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.

Since $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

~~P~~ Since $P(F) > 0$ and $P(E \cap F) \leq P(F)$, then (0,3)

$P(F) - P(E \cap F) \geq 0$ $\therefore P(E \cup F) \geq P(E) = 0.8$

Now, since $P(E \cup F) \leq 1$, we have $P(E) + P(F) - P(E \cap F) \leq 1$

$\therefore P(E \cap F) \geq P(E) + P(F) - 1 = 0.8 + 0.6 - 1 = 0.4$ (0,3)

$\therefore P(E \cap F) \geq 0.4$.

c. (5 points) What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

Let $E$ = Four heads appear in 5 flips of a coin.

$F$ = First flip in 5 flips of a coin is T.

$\therefore P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}$

Now, $P(E \cap F) = P(THHHH) = 1/2^5$. $P(F) = \dfrac{2^4}{2^5} = \dfrac{1}{2}$

$\therefore P(E \mid F) = \dfrac{1/2^5}{1/2} = \dfrac{1}{2^4}$.

(0,2,5)

d. (12 points) Find each of the following probabilities when *n* independent Bernoulli trials are carried out with probability of success p.

i. (3 points) the probability of no successes
ii. (3 points) the probability of at least one success
iii. (3 points) the probability of at most one success
iv. (3 points) the probability of at least two successes

(0,3)

i. $\binom{n}{0}(1-p)^n = (1-p)^n$

(0,3)

ii. $1 - (1-p)^n$.

iii. $(1-p)^n + \binom{n}{1} p(1-p)^{n-1} = (1-p)^n + np(1-p)^{n-1}$
$= (1-p)^{n-1}(1-p+np)$

(0,3)

iv. $1 - (1-p)^n - np(1-p)^{n-1}$

(0,3)