1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$. **b)** $a + b = 4$.
c) $a > b$. **d)** $a \mid b$.
e) $\gcd(a, b) = 1$. **f)** $\text{lcm}(a, b) = 2$.

1. In each case, we need to find all the pairs (a, b) with $a \in A$ and $b \in B$ such that the condition is satisfied. This is straightforward.

a) $\{(0,0), (1,1), (2,2), (3,3)\}$ **b)** $\{(1,3), (2,2), (3,1), (4,0)\}$

c) $\{(1,0), (2,0), (2,1), (3,0), (3,1), (3,2), (4,0), (4,1), (4,2), (4,3)\}$

d) Recall that $a|b$ means that b is a multiple of a (a is not allowed to be 0). Thus the answer is $\{(1,0), (1,1), (1,2), (1,3), (2,0), (2,2), (3,0), (3,3), (4,0)\}$.

e) We need to look for pairs whose greatest common divisor is 1—in other words, pairs that are relatively prime. Thus the answer is $\{(0,1), (1,0), (1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (4,1), (4,3)\}$.

f) There are not very many pairs of numbers (by definition only positive integers are considered) whose least common multiple is 2: only 1 and 2, and 2 and 2. Thus the answer is $\{(1,2), (2,1), (2,2)\}$.

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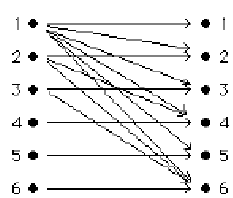
2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

2. a) $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,2)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,6)$, $(4,4)$, $(5,5)$, $(6,6)$

b) We draw a line from a to b whenever a divides b, using separate sets of points; an alternate form of this graph would have just one set of points.



c) We put an $\times$ in the i^{th} row and j^{th} column if and only if i divides j.

R	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

---

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

3. a) This relation is not reflexive, since it does not include, for instance $(1,1)$. It is not symmetric, since it includes, for instance, $(2,4)$ but not $(4,2)$. It is not antisymmetric since it includes both $(2,3)$ and $(3,2)$, but $2 \neq 3$. It is transitive. To see this we have to check that *whenever* it includes (a,b) and (b,c), then it also includes (a,c). We can ignore the element 1 since it never appears. If (a,b) is in this relation, then by inspection we see that a must be either 2 or 3. But $(2,c)$ and $(3,c)$ are in the relation for all $c \neq 1$; thus (a,c) has to be in this relation whenever (a,b) and (b,c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.

b) This relation is reflexive, since all the pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$ are in it. It is clearly symmetric, the only nontrivial case to note being that both $(1,2)$ and $(2,1)$ are in the relation. It is not antisymmetric because both $(1,2)$ and $(2,1)$ are in the relation. It is transitive; the only nontrivial cases to note are that since both $(1,2)$ and $(2,1)$ are in the relation, we need to have (and do have) both $(1,1)$ and $(2,2)$ included as well.

c) This relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both $(2,4)$ and $(4,2)$ are in the relation. It is not transitive, since although $(2,4)$ and $(4,2)$ are in the relation, $(2,2)$ is not.

d) This relation is clearly not reflexive. It is not symmetric, since, for instance, $(1,2)$ is included but $(2,1)$ is not. It is antisymmetric, since there are no cases of (a,b) and (b,a) both being in the relation. It is not transitive, since although $(1,2)$ and $(2,3)$ are in the relation, $(1,3)$ is not.

e) This relation is clearly reflexive and symmetric. It is trivially antisymmetric since there are no pairs (a,b) in the relation with $a \neq b$. It is trivially transitive, since the only time the hypothesis $(a,b) \in R \wedge (b,c) \in R$ is met is when $a = b = c$.

f) This relation is clearly not reflexive. The presence of $(1,4)$ and absence of $(4,1)$ shows that it is not symmetric. The presence of both $(1,3)$ and $(3,1)$ shows that it is not antisymmetric. It is not transitive; both $(2,3)$ and $(3,1)$ are in the relation, but $(2,1)$ is not, for instance.

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.

b) a and b were born on the same day.

c) a has the same first name as b.

d) a and b have a common grandparent.

4. a) Being taller than is not reflexive (I am not taller than myself), nor symmetric (I am taller than my daughter, but she is not taller than I). It is antisymmetric (vacuously, since we never have A taller than B, and B taller than A, even if $A = B$). It is clearly transitive.

b) This is clearly reflexive, symmetric, and transitive (it is an equivalence relation—see Section 9.5). It is not antisymmetric, since twins, for example, are unequal people born on the same day.

c) This has exactly the same answers as part **(b)**, since having the same first name is just like having the same birthday.

d) This is clearly reflexive and symmetric. It is not antisymmetric, since my cousin and I have a common grandparent, and I and my cousin have a common grandparent, but I am not equal to my cousin. This relation is not transitive. My cousin and I have a common grandparent; my cousin and her cousin on the other side of her family have a common grandparent. My cousin's cousin and I do not have a common grandparent.

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5. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) everyone who has visited Web page a has also visited Web page b.

b) there are no common links found on both Web page a and Web page b.

c) there is at least one common link on Web page a and Web page b.

d) there is a Web page that includes links to both Web page a and Web page b.

5. Recall the definitions: R is reflexive if $(a,a) \in R$ for all a; R is symmetric if $(a,b) \in R$ always implies $(b,a) \in R$; R is antisymmetric if $(a,b) \in R$ and $(b,a) \in R$ always implies $a = b$; and R is transitive if $(a,b) \in R$ and $(b,c) \in R$ always implies $(a,c) \in R$.

a) It is tautological that everyone who has visited Web page a has also visited Web page a, so R is reflexive. It is not symmetric, because there surely are Web pages a and b such that the set of people who visited a is a proper subset of the set of people who visited b (for example, the only link to page a may be on page b). Whether R is antisymmetric in truth is hard to say, but it is certainly conceivable that there are two different Web pages a and b that have had exactly the same set of visitors. In this case, $(a,b) \in R$ and $(b,a) \in R$, so R is not antisymmetric. Finally, R is transitive: if everyone who has visited a has also visited b, and everyone who has visited b has also visited c, then clearly everyone who has visited a has also visited c.

b) This relation is not reflexive, because for any page a that has links on it, $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have no common links found on them. Finally, R is not transitive, because the two Web pages just mentioned, assuming they have links at all, give an example of the failure of the definition: $(a,b) \in R$ and $(b,a) \in R$, but $(a,a) \notin R$.

c) This relation is not reflexive, because for any page a that has no links on it, $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have a common link found on them. Finally, R is surely not transitive. Page a might have only one link (say to this textbook), page c might have only one link different from this (say to the Erdős Number Project), and page b may have only the two links mentioned in this sentence. Then $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$.

d) This relation is probably not reflexive, because there probably exist Web pages out there with no links at all to them (for example, when they are in the process of being written and tested); for any such page a we have $(a,a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that are referenced by some third page. Finally, R is surely not transitive. Page a might have only one page that links to it, page c might also have only one page, different from this, that links to it, and page b may be cited on both of these two pages. Then there would be no page that includes links to both pages a and c, so we have $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$.

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$.
b) $x = \pm y$.
c) $x - y$ is a rational number.
d) $x = 2y$.
e) $xy \geq 0$.
f) $xy = 0$.
g) $x = 1$.
h) $x = 1$ or $y = 1$.

6. a) Since $1 + 1 \neq 0$, this relation is not reflexive. Since $x + y = y + x$, it follows that $x + y = 0$ if and only if $y + x = 0$, so the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1, -1) \in R$ and $(-1, 1) \in R$, but $(1, 1) \notin R$.

b) Since $x = \pm x$ (choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1.

c) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x - y) = y - x$. Since $(1, -1)$ and $(-1, 1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$.

d) Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2, 1) \in R$, but $(1, 2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x, y) and (y, x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4, 2) \in R$ and $(2, 1) \in R$, but $(4, 1) \notin R$.

e) This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 3)$ and $(3, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

f) This is not reflexive, since $(1, 1) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 0)$ and $(0, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

g) This is not reflexive, since $(2, 2) \notin R$. It is not symmetric, since $(1, 2) \in R$ but $(2, 1) \notin R$. It is antisymmetric, because if $(x, y) \in R$ and $(y, x) \in R$, then $x = 1$ and $y = 1$, so $x = y$. It is transitive, because if $(x, y) \in R$ and $(y, z) \in R$, then $x = 1$ (and $y = 1$, although that doesn't matter), so $(x, z) \in R$.

h) This is not reflexive, since $(2, 2) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 1)$ and $(1, 2)$ are both in R. It is not transitive; for example, $(3, 1) \in R$ and $(1, 7) \in R$, but $(3, 7) \notin R$.

7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$. **b)** $xy \geq 1$.
c) $x = y + 1$ or $x = y - 1$.
d) $x \equiv y \pmod 7$. **e)** x is a multiple of y.
f) x and y are both negative or both nonnegative.
g) $x = y^2$. **h)** $x \geq y^2$.

7. a) This relation is not reflexive since it is not the case that $1 \neq 1$, for instance. It is symmetric: if $x \neq y$, then of course $y \neq x$. It is not antisymmetric, since, for instance, $1 \neq 2$ and also $2 \neq 1$. It is not transitive, since $1 \neq 2$ and $2 \neq 1$, for instance, but it is not the case that $1 \neq 1$.

b) This relation is not reflexive, since $(0,0)$ is not included. It is symmetric, because the commutative property of multiplication tells us that $xy = yx$, so that one of these quantities is greater than or equal to 1 if and only if the other is. It is not antisymmetric, since, for instance, $(2,3)$ and $(3,2)$ are both included. It is transitive. To see this, note that the relation holds between x and y if and only if either x and y are both positive or x and y are both negative. So assume that (a,b) and (b,c) are both in the relation. There are two cases, nearly identical. If a is positive, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$. If a is negative, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$.

c) This relation is not reflexive, since $(1,1)$ is not included, for instance. It is symmetric; the equation $x = y - 1$ is equivalent to the equation $y = x + 1$, which is the same as the equation $x = y + 1$ with the roles of x and y reversed. (A more formal proof of symmetry would be by cases. If x is related to y then either $x = y + 1$ or $x = y - 1$. In the former case, $y = x - 1$, so y is related to x; in the latter case $y = x + 1$, so y is related to x.) It is not antisymmetric, since, for instance, both $(1,2)$ and $(2,1)$ are in the relation. It is not transitive, since, for instance, although both $(1,2)$ and $(2,1)$ are in the relation, $(1,1)$ is not.

d) Recall that $x \equiv y \pmod 7$ means that $x - y$ is a multiple of 7, i.e., that $x - y = 7t$ for some integer t. This relation is reflexive, since $x - x = 7 \cdot 0$ for all x. It is symmetric, since if $x \equiv y \pmod 7$, then $x - y = 7t$ for some t; therefore $y - x = 7(-t)$, so $y \equiv x \pmod 7$. It is not antisymmetric, since, for instance, we have both $2 \equiv 9$ and $9 \equiv 2 \pmod 7$. It is transitive. Suppose $x \equiv y$ and $y \equiv z \pmod 7$. This means that $x - y = 7s$ and $y - z = 7t$ for some integers s and t. The trick is to add these two equations and note that the y disappears; we get $x - z = 7s + 7t = 7(s+t)$. By definition, this means that $x \equiv z \pmod 7$, as desired.

e) Every number is a multiple of itself (namely 1 times itself), so this relation is reflexive. (There is one bit of controversy here; we assume that 0 is to be considered a multiple of 0, even though we do not consider that 0 is a divisor of 0.) It is clearly not symmetric, since, for instance, 6 is a multiple of 2, but 2 is not a multiple of 6. The relation is not antisymmetric either; we have that 2 is a multiple of -2, for instance, and -2 is a multiple of 2, but $2 \neq -2$. The relation is transitive, however. If x is a multiple of y (say $x = ty$), and y is a multiple of z (say $y = sz$), then we have $x = t(sz) = (ts)z$, so we know that x is a multiple of z.

f) This relation is reflexive, since a and a are either both negative or both nonnegative. It is clearly symmetric from its form. It is not antisymmetric, since 5 is related to 6 and 6 is related to 5, but $5 \neq 6$. Finally, it is transitive, since if a is related to b and b is related to c, then all three of them must be negative, or all three must be nonnegative.

8. Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.

8. If $R = \emptyset$, then the hypotheses of the conditional statements in the definitions of *symmetric* and *transitive* are never true, so those statements are always true by definition. Because $S \neq \emptyset$, the statement $(a, a) \in R$ is false for an element of S, so $\forall a\, (a, a) \in R$ is not true; thus R is not reflexive.

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9. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

9. Each of the properties is a universally quantified statement. Because the domain is empty, each of them is vacuously true.

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10. Give an example of a relation on a set that is

a) both symmetric and antisymmetric.

b) neither symmetric nor antisymmetric.

A relation R on the set A is **irreflexive** if for every $a \in A$, $(a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

10. We give the simplest example in each case.

a) the empty set on $\{a\}$ (vacuously symmetric and antisymmetric)

b) $\{(a, b), (b, a), (a, c)\}$ on $\{a, b, c\}$

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15. Can a relation on a set be neither reflexive nor irreflexive?

15. The relation in Exercise 3a is neither reflexive nor irreflexive. It contains some of the pairs (a, a) but not all of them.

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16. Use quantifiers to express what it means for a relation to be irreflexive.

16. $\forall x\, ((x, x) \notin R)$

17. Give an example of an irreflexive relation on the set of all people.

17. Of course many answers are possible. The empty relation is always irreflexive (x is never related to y). A less trivial example would be $(a, b) \in R$ if and only if a is taller than b. Since nobody is taller than him/herself, we always have $(a, a) \notin R$.

A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

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22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.

22. An asymmetric relation must be antisymmetric, since the hypothesis of the condition for antisymmetry is false if the relation is asymmetric. The relation $\{(a, a)\}$ on $\{a\}$ is antisymmetric but not asymmetric, however, so the answer to the second question is no. In fact, it is easy to see that R is asymmetric if and only if R is antisymmetric and irreflexive.

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25. How many different relations are there from a set with m elements to a set with n elements?

25. There are mn elements of the set $A \times B$, if A is a set with m elements and B is a set with n elements. A relation from A to B is a subset of $A \times B$. Thus the question asks for the number of subsets of the set $A \times B$, which has mn elements. By the product rule, it is 2^{mn}.

☞ Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1}. **b)** $\overline{R}$.

26. a) $R^{-1} = \{\, (b,a) \mid (a,b) \in R \,\} = \{\, (b,a) \mid a < b \,\} = \{\, (a,b) \mid a > b \,\}$

b) $\overline{R} = \{\, (a,b) \mid (a,b) \notin R \,\} = \{\, (a,b) \mid a \not< b \,\} = \{\, (a,b) \mid a \geq b \,\}$

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27. Let R be the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integers. Find

a) R^{-1}. **b)** $\overline{R}$.

27. a) By definition the answer is $\{\, (b,a) \mid a \text{ divides } b \,\}$, which, by changing the names of the dummy variables, can also be written $\{\, (a,b) \mid b \text{ divides } a \,\}$. (The universal set is still the set of positive integers.)

b) By definition the answer is $\{\, (a,b) \mid a \text{ does not divide } b \,\}$. (The universal set is still the set of positive integers.)

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28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1}. **b)** $\overline{R}$.

28. a) Since this relation is symmetric, $R^{-1} = R$.

b) This relation consists of all pairs (a,b) in which state a does not border state b.

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29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1}?

29. The inverse relation is just the graph of the inverse function. Somewhat more formally, we have $R^{-1} = \{\, (f(a), a) \mid a \in A \,\} = \{\, (b, f^{-1}(b)) \mid b \in B \,\}$, since we can index this collection just as easily by elements of B as by elements of A (using the correspondence $b = f(a)$).

30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

a) $R_1 \cup R_2$.

b) $R_1 \cap R_2$.

c) $R_1 - R_2$.

d) $R_2 - R_1$.

30. These are merely routine exercises in set theory. Note that $R_1 \subseteq R_2$.

a) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\} = R_2$ **b)** $\{(1,2),(2,3),(3,4)\} = R_1$

c) $\emptyset$ **d)** $\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

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31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.

a) $R_1 \cup R_2$

b) $R_1 \cap R_2$

c) $R_1 \oplus R_2$

d) $R_1 - R_2$

e) $R_2 - R_1$

31. This exercise is just a matter of the definitions of the set operations.

a) the set of pairs (a,b) where a is required to read b in a course or has read b

b) the set of pairs (a,b) where a is required to read b in a course and has read b

c) the set of pairs (a,b) where a is required to read b in a course or has read b, but not both; equivalently, the set of pairs (a,b) where a is required to read b in a course but has not done so, or has read b although not required to do so in a course

d) the set of pairs (a,b) where a is required to read b in a course but has not done so

e) the set of pairs (a,b) where a has read b although not required to do so in a course

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32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

32. Since $(1,2) \in R$ and $(2,1) \in S$, we have $(1,1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1,1),(1,2),(2,1),(2,2)\}$.

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33. Let R be the relation on the set of people consisting of pairs (a, b), where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a, b), where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

33. To find $S \circ R$ we want to find the set of pairs (a, c) such that for some person b, a is a parent of b, and b is a sibling of c. Since brothers and sisters have the same parents, this means that a is also the parent of c. Thus $S \circ R$ is contained in the relation R. More specifically, $(a, c) \in S \circ R$ if and only if a is the parent of c, and c has a sibling (who is necessarily also a child of a). To find $R \circ S$ we want to find the set of pairs (a, c) such that for some person b, a is a sibling of b, and b is a parent of c. This is the same as the condition that a is the aunt or uncle of c (by blood, not marriage).

Exercises 34–37 deal with these relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the "greater than" relation,

$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the "greater than or equal to" relation,

$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the "less than" relation,

$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the "less than or equal to" relation,

$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the "equal to" relation,

$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the "unequal to" relation.

34. Find

a) $R_1 \cup R_3$.
b) $R_1 \cup R_5$.
c) $R_2 \cap R_4$.
d) $R_3 \cap R_5$.
e) $R_1 - R_2$.
f) $R_2 - R_1$.
g) $R_1 \oplus R_3$.
h) $R_2 \oplus R_4$.

34. a) The union of two relations is the union of these sets. Thus $R_1 \cup R_3$ holds between two real numbers if R_1 holds or R_3 holds (or both, it goes without saying). Here this means that the first number is greater than the second or vice versa—in other words, that the two numbers are not equal. This is just relation R_6.

b) For (a, b) to be in $R_3 \cup R_6$, we must have $a > b$ or $a = b$. Since this happens precisely when $a \geq b$, we see that the answer is R_2.

c) The intersection of two relations is the intersection of these sets. Thus $R_2 \cap R_4$ holds between two real numbers if R_2 holds and R_4 holds as well. Thus for (a, b) to be in $R_2 \cap R_4$, we must have $a \geq b$ and $a \leq b$. Since this happens precisely when $a = b$, we see that the answer is R_5.

d) For (a, b) to be in $R_3 \cap R_5$, we must have $a < b$ and $a = b$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

e) Recall that $R_1 - R_2 = R_1 \cap \overline{R_2}$. But $\overline{R_2} = R_3$, so we are asked for $R_1 \cap R_3$. It is impossible for $a > b$ and $a < b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

f) Reasoning as in part (f), we want $R_2 \cap \overline{R_1} = R_2 \cap R_4$, which is R_5 (this was part (c)).

g) Recall that $R_1 \oplus R_3 = (R_1 \cap \overline{R_3}) \cup (R_3 \cap \overline{R_1})$. We see that $R_1 \cap \overline{R_3} = R_1 \cap R_2 = R_1$, and $R_3 \cap \overline{R_1} = R_3 \cap R_4 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).

h) Recall that $R_2 \oplus R_4 = (R_2 \cap \overline{R_4}) \cup (R_4 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_4} = R_2 \cap R_1 = R_1$, and $R_4 \cap \overline{R_2} = R_4 \cap R_3 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).

35. Find

a) $R_2 \cup R_4$.
b) $R_3 \cup R_6$.
c) $R_3 \cap R_6$.
d) $R_4 \cap R_6$.
e) $R_3 - R_6$.
f) $R_6 - R_3$.
g) $R_2 \oplus R_6$.
h) $R_3 \oplus R_5$.

35. a) The union of two relations is the union of these sets. Thus $R_2 \cup R_4$ holds between two real numbers if R_2 holds or R_4 holds (or both, it goes without saying). Since it is always true that $a \leq b$ or $b \leq a$, $R_2 \cup R_4$ is all of $\mathbf{R}^2$, i.e., the relation that always holds.

b) For (a, b) to be in $R_3 \cup R_6$, we must have $a < b$ or $a \neq b$. Since this happens precisely when $a \neq b$, we see that the answer is R_6.

c) The intersection of two relations is the intersection of these sets. Thus $R_3 \cap R_6$ holds between two real numbers if R_3 holds and R_6 holds as well. Thus for (a, b) to be in $R_3 \cap R_6$, we must have $a < b$ and $a \neq b$. Since this happens precisely when $a < b$, we see that the answer is R_3.

d) For (a, b) to be in $R_4 \cap R_6$, we must have $a \leq b$ and $a \neq b$. Since this happens precisely when $a < b$, we see that the answer is R_3.

e) Recall that $R_3 - R_6 = R_3 \cap \overline{R_6}$. But $\overline{R_6} = R_5$, so we are asked for $R_3 \cap R_5$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

f) Reasoning as in part **(e)**, we want $R_6 \cap \overline{R_3} = R_6 \cap R_2$, which is clearly R_1 (since $a \neq b$ and $a \geq b$ precisely when $a > b$).

g) Recall that $R_2 \oplus R_6 = (R_2 \cap \overline{R_6}) \cup (R_6 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_6} = R_2 \cap R_5 = R_5$, and $R_6 \cap \overline{R_2} = R_6 \cap R_3 = R_3$. Thus our answer is $R_5 \cup R_3 = R_4$.

h) Recall that $R_3 \oplus R_5 = (R_3 \cap \overline{R_5}) \cup (R_5 \cap \overline{R_3})$. We see that $R_3 \cap \overline{R_5} = R_3 \cap R_6 = R_3$, and $R_5 \cap \overline{R_3} = R_5 \cap R_2 = R_5$. Thus our answer is $R_3 \cup R_5 = R_4$.

36. Find

a) $R_1 \circ R_1$.
b) $R_1 \circ R_2$.
c) $R_1 \circ R_3$.
d) $R_1 \circ R_4$.
e) $R_1 \circ R_5$.
f) $R_1 \circ R_6$.
g) $R_2 \circ R_3$.
h) $R_3 \circ R_3$.

36. Recall that the composition of two relations all defined on a common set is defined as follows: $(a,c) \in S \circ R$ if and only if there is some element b such that $(a,b) \in R$ and $(b,c) \in S$. We have to apply this in each case.

a) For (a,c) to be in $R_1 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_1$. This means that $a > b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_1 = R_1$. We can interpret (part of) this as showing that R_1 is transitive.

b) For (a,c) to be in $R_1 \circ R_2$, we must find an element b such that $(a,b) \in R_2$ and $(b,c) \in R_1$. This means that $a \geq b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_2 = R_1$.

c) For (a,c) to be in $R_1 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_1$. This means that $a < b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

d) For (a,c) to be in $R_1 \circ R_4$, we must find an element b such that $(a,b) \in R_4$ and $(b,c) \in R_1$. This means that $a \leq b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_4 = \mathbf{R}^2$, the relation that always holds.

e) For (a,c) to be in $R_1 \circ R_5$, we must find an element b such that $(a,b) \in R_5$ and $(b,c) \in R_1$. This means that $a = b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with (choose $b = a$). But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_5 = R_1$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).

f) For (a,c) to be in $R_1 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_1$. This means that $a \neq b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

g) For (a,c) to be in $R_2 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_2$. This means that $a < b$ and $b \geq c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_2 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

h) For (a,c) to be in $R_3 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_3$. This means that $a < b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with. But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_3 \circ R_3 = R_3$. We can interpret (part of) this as showing that R_3 is transitive.

37. Find

a) $R_2 \circ R_1$.
b) $R_2 \circ R_2$.
c) $R_3 \circ R_5$.
d) $R_4 \circ R_1$.
e) $R_5 \circ R_3$.
f) $R_3 \circ R_6$.
g) $R_4 \circ R_6$.
h) $R_6 \circ R_6$.

37. Recall that the composition of two relations all defined on a common set is defined as follows: $(a, c) \in S \circ R$ if and only if there is some element b such that $(a, b) \in R$ and $(b, c) \in S$. We have to apply this in each case.

a) For (a, c) to be in $R_2 \circ R_1$, we must find an element b such that $(a, b) \in R_1$ and $(b, c) \in R_2$. This means that $a > b$ and $b \geq c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a, c) \in R_1$. Therefore we have $R_2 \circ R_1 = R_1$.

b) For (a, c) to be in $R_2 \circ R_2$, we must find an element b such that $(a, b) \in R_2$ and $(b, c) \in R_2$. This means that $a \geq b$ and $b \geq c$. Clearly this can be done if and only if $a \geq c$ to begin with. But that is precisely the statement that $(a, c) \in R_2$. Therefore we have $R_2 \circ R_2 = R_2$. In particular, this shows that R_2 is transitive.

c) For (a, c) to be in $R_3 \circ R_5$, we must find an element b such that $(a, b) \in R_5$ and $(b, c) \in R_3$. This means that $a = b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = a$). But that is precisely the statement that $(a, c) \in R_3$. Therefore we have $R_3 \circ R_5 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).

d) For (a, c) to be in $R_4 \circ R_1$, we must find an element b such that $(a, b) \in R_1$ and $(b, c) \in R_4$. This means that $a > b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_1 = \mathbf{R}^2$, the relation that always holds.

e) For (a, c) to be in $R_5 \circ R_3$, we must find an element b such that $(a, b) \in R_3$ and $(b, c) \in R_5$. This means that $a < b$ and $b = c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = c$). But that is precisely the statement that $(a, c) \in R_3$. Therefore we have $R_5 \circ R_3 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the left—although it is also an identity on the right as well).

f) For (a, c) to be in $R_3 \circ R_6$, we must find an element b such that $(a, b) \in R_6$ and $(b, c) \in R_3$. This means that $a \neq b$ and $b < c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_3 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

g) For (a, c) to be in $R_4 \circ R_6$, we must find an element b such that $(a, b) \in R_6$ and $(b, c) \in R_4$. This means that $a \neq b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

h) For (a, c) to be in $R_6 \circ R_6$, we must find an element b such that $(a, b) \in R_6$ and $(b, c) \in R_6$. This means that $a \neq b$ and $b \neq c$. Clearly this can always be done simply by choosing b to be something other than a or c. Therefore we have $R_6 \circ R_6 = \mathbf{R}^2$, the relation that always holds. Note that since the answer is not R_6 itself, we know that R_6 is not transitive.

40. Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a$ is a multiple of $b\}$. Find

a) $R_1 \cup R_2$.
b) $R_1 \cap R_2$.
c) $R_1 - R_2$.
d) $R_2 - R_1$.
e) $R_1 \oplus R_2$.

40. Note that these two relations are inverses of each other, since a is a multiple of b if and only if b divides a (see the preamble to Exercise 26).

a) The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a, b) \in R_1 \cup R_2$ if and only if $a \mid b$ or $b \mid a$. There is not a good easier way to state this.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a, b) \in R_1 \cap R_2$ if and only if $a \mid b$ and $b \mid a$. This happens if and only if $a = \pm b$ and $a \neq 0$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a, b) \in R_1 - R_2$ if and only if $a \mid b$ and $b \nmid a$. This is equivalent to saying that $a \mid b$ and $a \neq \pm b$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a, b) \in R_2 - R_1$ if and only if $b \mid a$ and $a \nmid b$. This is equivalent to saying that $b \mid a$ and $a \neq \pm b$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part **(c)** and part **(d)**. Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. This happens if and only if $a \mid b$ or $b \mid a$, but $a \neq \pm b$.

41. Let R_1 and R_2 be the "congruent modulo 3" and the "congruent modulo 4" relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod 3\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod 4\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.
e) $R_1 \oplus R_2$.

41. a) The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a, b) \in R_1 \cup R_2$ if and only if $a \equiv b \pmod 3$ or $a \equiv b \pmod 4$. There is not a good easier way to state this, other than perhaps to say that $a - b$ is a multiple of either 3 or 4, or to work modulo 12 and write $a - b \equiv 0, 3, 4, 6, 8,$ or $9 \pmod{12}$.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a, b) \in R_1 \cap R_2$ if and only if $a \equiv b \pmod 3$ and $a \equiv b \pmod 4$. Since this means that $a - b$ is a multiple of both 3 and 4, and that happens if and only if $a - b$ is a multiple of 12, we can state this more simply as $a \equiv b \pmod{12}$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a, b) \in R_1 - R_2$ if and only if $a \equiv b \pmod 3$ and $a \not\equiv b \pmod 4$. We could, if we wished, state this working modulo 12: $(a, b) \in R_1 - R_2$ if and only if $a - b \equiv 3, 6,$ or $9 \pmod{12}$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a, b) \in R_2 - R_1$ if and only if $a \equiv b \pmod 4$ and $a \not\equiv b \pmod 3$. We could, if we wished, state this working modulo 12: $(a, b) \in R_2 - R_1$ if and only if $a - b \equiv 4$ or $8 \pmod{12}$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part **(c)** and part **(d)**. Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. We can write this in symbols by saying that $(a, b) \in R_1 \oplus R_2$ if and only if $(a \equiv b \pmod 3$ and $a \not\equiv b \pmod 4)$ or $(a \equiv b \pmod 4$ and $a \not\equiv b \pmod 3)$. We could, if we wished, state this working modulo 12: $(a, b) \in R_1 \oplus R_2$ if and only if $a - b \equiv 3, 4, 6, 8$ or $9 \pmod{12}$. We could also say that $a - b$ is a multiple of 3 or 4 but not both.

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42. List the 16 different relations on the set $\{0, 1\}$.

42. These are just the 16 different subsets of $\{(0,0), (0,1), (1,0), (1,1)\}$.

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0), (0,1)\}$
7. $\{(0,0), (1,0)\}$
8. $\{(0,0), (1,1)\}$
9. $\{(0,1), (1,0)\}$
10. $\{(0,1), (1,1)\}$
11. $\{(1,0), (1,1)\}$
12. $\{(0,0), (0,1), (1,0)\}$
13. $\{(0,0), (0,1), (1,1)\}$
14. $\{(0,0), (1,0), (1,1)\}$
15. $\{(0,1), (1,0), (1,1)\}$
16. $\{(0,0), (0,1), (1,0), (1,1)\}$

43. How many of the 16 different relations on {0, 1} contain the pair (0, 1)?

43. A relation is just a subset. A subset can either contain a specified element or not; half of them do and half of them do not. Therefore 8 of the 16 relations on $\{0,1\}$ contain the pair $(0,1)$. Alternatively, a relation on $\{0,1\}$ containing the pair $(0,1)$ is just a set of the form $\{(0,1)\} \cup X$, where $X \subseteq \{(0,0),(1,0),(1,1)\}$. Since this latter set has 3 elements, it has $2^3 = 8$ subsets.

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44. Which of the 16 relations on {0, 1}, which you listed in Exercise 42, are

a) reflexive? **b)** irreflexive?
c) symmetric? **d)** antisymmetric?
e) asymmetric? **f)** transitive?

44. We list the relations by number as given in the solution above.

a) 8, 13, 14, 16 b) 1, 3, 4, 9 c) 1, 2, 5, 8, 9, 12, 15, 16

d) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14 e) 1, 3, 4 f) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16

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45. a) How many relations are there on the set $\{a, b, c, d\}$?
b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a)?

45. This is similar to Example 16 in this section.

a) A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, we are asking for the number of subsets of a set with n^2 elements, which is 2^{n^2}. In our case $n = 4$, so the answer is $2^{16} = 65{,}536$.

b) In solving part **(a)**, we had 16 binary choices to make—whether to include a pair (x,y) in the relation or not as x and y ranged over the set $\{a,b,c,d\}$. In this part, one of those choices has been made for us: we *must* include (a,a). We are free to make the other 15 choices. So the answer is $2^{15} = 32{,}768$. See Exercise 47 for more problems of this type.

46. Let S be a set with n elements and let a and b be distinct elements of S. How many relations R are there on S such that

a) $(a, b) \in R$?
b) $(a, b) \notin R$?
c) no ordered pair in R has a as its first element?
d) at least one ordered pair in R has a as its first element?
e) no ordered pair in R has a as its first element or b as its second element?
f) at least one ordered pair in R either has a as its first element or has b as its second element?

46. This is similar to Example 16 in this section. A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, so there are 2^{n^2} relations on S if no restrictions are imposed. One might observe here that the condition that $a \neq b$ is not relevant.

a) Half of these relations contain (a, b) and half do not, so the answer is $2^{n^2}/2 = 2^{n^2-1}$. Looking at it another way, we see that there are $n^2 - 1$ choices involved in specifying such a relation, since we have no choice about (a, b).

b) The analysis and answer are exactly the same as in part **(a)**.

c) Of the n^2 possible pairs to put in R, exactly n of them have a as their first element. We must use none of these, so there are $n^2 - n$ pairs that we are free to work with. Therefore there are 2^{n^2-n} possible choices for R.

d) By part **(c)** we know that there are 2^{n^2-n} relations that do not contain at least one ordered pair with a as its first element, so all the other relations, namely $2^{n^2} - 2^{n^2-n}$ of them, do contain at least one ordered pair with a as its first element.

e) We reason as in part **(c)**. There are n ordered pairs that have a as their first element, and n more that have b as their second element, although this counts (a, b) twice, so there are a total of $2n - 1$ pairs that violate the condition. This means that there are $n^2 - 2n + 1 = (n-1)^2$ pairs that we are free to choose for R. Thus the answer is $2^{(n-1)^2}$. Another way to look at this is to visualize the matrix representing R. The a^{th} row must be all 0's, as must the b^{th} column. If we cross out that row and column we have in effect an $n - 1$ by $n - 1$ matrix, with $(n-1)^2$ entries. Since we can fill each entry with either a 0 or a 1, there are $2^{(n-1)^2}$ choices for specifying S.

f) This is the opposite condition from part **(e)**. Therefore reasoning as in part **(d)**, we have $2^{n^2} - 2^{(n-1)^2}$ possible relations.

***47.** How many relations are there on a set with n elements that are

a) symmetric? **b)** antisymmetric?
c) asymmetric? **d)** irreflexive?
e) reflexive and symmetric?
f) neither reflexive nor irreflexive?

47. These are combinatorics problems, some harder than others. Let A be the set with n elements on which the relations are defined.

a) To specify a symmetric relation, we need to decide, for each unordered pair $\{a, b\}$ of distinct elements of A, whether to include the pairs (a, b) and (b, a) or leave them out; this can be done in 2 ways for each such unordered pair. Also, for each element $a \in A$, we need to decide whether to include (a, a) or not, again 2 possibilities. We can think of these two parts as one by considering an element to be an unordered pair with repetition allowed. Thus we need to make this 2-fold choice $C(n+1, 2)$ times, since there are $C(n+2-1, 2)$ ways to choose an unordered pair with repetition allowed. Therefore the answer is $2^{C(n+1,2)} = 2^{n(n+1)/2}$.

***48.** How many transitive relations are there on a set with n elements if

a) $n = 1$? **b)** $n = 2$? **c)** $n = 3$?

48. **a)** There are two relations on a set with only one element, and they are both transitive.

b) There are 16 relations on a set with two elements, and we saw in Exercise 42f that 13 of them are transitive.

c) For $n = 3$ there are $2^{3^2} = 512$ relations. One way to find out how many of them are transitive is to use a computer to generate them all and check each one for transitivity. If we do this, then we find that 171 of them are transitive. Doing this by hand is not pleasant, since there are many cases to consider.

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50. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive.
b) $R \cap S$ is reflexive.
c) $R \oplus S$ is irreflexive.
d) $R - S$ is irreflexive.
e) $S \circ R$ is reflexive.

50. a) Since R contains all the pairs (x, x), so does $R \cup S$. Therefore $R \cup S$ is reflexive.

b) Since R and S each contain all the pairs (x, x), so does $R \cap S$. Therefore $R \cap S$ is reflexive.

c) Since R and S each contain all the pairs (x, x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x, x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

e) Since R and S each contain all the pairs (x, x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

---

51. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.

51. We need to show two things. First, we need to show that if a relation R is symmetric, then $R = R^{-1}$, which means we must show that $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$. To do this, let $(a, b) \in R$. Since R is symmetric, this implies that $(b, a) \in R$. But since R^{-1} consists of all pairs (a, b) such that $(b, a) \in R$, this means that $(a, b) \in R^{-1}$. Thus we have shown that $R \subseteq R^{-1}$. Next let $(a, b) \in R^{-1}$. By definition this means that $(b, a) \in R$. Since R is symmetric, this implies that $(a, b) \in R$ as well. Thus we have shown that $R^{-1} \subseteq R$.

Second we need to show that $R = R^{-1}$ implies that R is symmetric. To this end we let $(a, b) \in R$ and try to show that (b, a) is also necessarily an element of R. Since $(a, b) \in R$, the definition tells us that $(b, a) \in R^{-1}$. But since we are under the hypothesis that $R = R^{-1}$, this tells us that $(b, a) \in R$, exactly as desired.

---

52. Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

52. By definition, to say that R is antisymmetric is to say that $R \cap R^{-1}$ contains only pairs of the form (a, a). The statement we are asked to prove is just a rephrasing of this.

---

53. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.

53. Suppose that R is reflexive. We must show that R^{-1} is reflexive, i.e., that $(a, a) \in R^{-1}$ for each $a \in A$. Now since R is reflexive, we know that $(a, a) \in R$ for each $a \in R$. By definition, this tells us that $(a, a) \in R^{-1}$, as desired. (Interchanging the two a's in the pair (a, a) leaves it as it was.) Conversely, if R^{-1} is reflexive, then $(a, a) \in R^{-1}$ for each $a \in A$. By definition this means that $(a, a) \in R$ (again we interchanged the two a's).

---

54. Show that the relation R on a set A is reflexive if and only if the complementary relation $\overline{R}$ is irreflexive.

54. This is immediate from the definition, since R is reflexive if and only if it contains all the pairs (x, x), which in turn happens if and only if $\overline{R}$ contains none of these pairs, i.e., $\overline{R}$ is irreflexive.

---

55. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

55. We prove this by induction on n. The case $n = 1$ is trivial, since it is the statement $R = R$. Assume the inductive hypothesis that $R^n = R$. We must show that $R^{n+1} = R$. By definition $R^{n+1} = R^n \circ R$. Thus our task is to show that $R^n \circ R \subseteq R$ and $R \subseteq R^n \circ R$. The first uses the transitivity of R, as follows. Suppose that $(a, c) \in R^n \circ R$. This means that there is an element b such that $(a, b) \in R$ and $(b, c) \in R^n$. By the inductive hypothesis, the latter statement implies that $(b, c) \in R$. Thus by the transitivity of R, we know that $(a, c) \in R$, as desired.

Next assume that $(a, b) \in R$. We must show that $(a, b) \in R^n \circ R$. By the inductive hypothesis, $R^n = R$, and therefore R^n is reflexive by assumption. Thus $(b, b) \in R^n$. Since we have $(a, b) \in R$ and $(b, b) \in R^n$, we have by definition that (a, b) is an element of $R^n \circ R$, exactly as desired. (The first half of this proof was not really necessary, since Theorem 1 in this section already told us that $R^n \subseteq R$ for all n.)

---

56. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 4)$, $(3, 5)$, $(4, 2)$, $(4, 5)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4. **d)** R^5.

56. We just apply the definition each time. We find that R^2 contains all the pairs in $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ except $(2, 3)$ and $(4, 5)$; and R^3, R^4, and R^5 contain all the pairs.

57. Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integers n.

57. We use induction on n, the result being trivially true for $n = 1$. Assume that R^n is reflexive; we must show that R^{n+1} is reflexive. Let $a \in A$, where A is the set on which R is defined. By definition $R^{n+1} = R^n \circ R$. By the inductive hypothesis, R^n is reflexive, so $(a, a) \in R^n$. Also, since R is reflexive by assumption, $(a, a) \in R$. Therefore by the definition of composition, $(a, a) \in R^n \circ R$, as desired.

***58.** Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.

58. We prove this by induction on n. There is nothing to prove in the basis step ($n = 1$). Assume the inductive hypothesis that R^n is symmetric, and let $(a, c) \in R^{n+1} = R^n \circ R$. Then there is a $b \in A$ such that $(a, b) \in R$ and $(b, c) \in R^n$. Since R^n and R are symmetric, $(b, a) \in R$ and $(c, b) \in R^n$. Thus by definition $(c, a) \in R \circ R^n$. We will have completed the proof if we can show that $R \circ R^n = R^{n+1}$. This we do in two steps. First, composition of relations is associative, that is, $(R \circ S) \circ T = R \circ (S \circ T)$ for all relations with appropriate domains and codomains. (The proof of this is straightforward applications of the definition.) Second we show that $R \circ R^n = R^{n+1}$ by induction on n. Again the basis step is trivial. Under the inductive hypothesis, then, $R \circ R^{n+1} = R \circ (R^n \circ R) = (R \circ R^n) \circ R = R^{n+1} \circ R = R^{n+2}$, as desired.

---

59. Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

59. It is not necessarily true that R^2 is irreflexive when R is. We might have pairs (a, b) and (b, a) both in R, with $a \neq b$; then it would follow that $(a, a) \in R^2$, preventing R^2 from being irreflexive. As the simplest example, let $A = \{1, 2\}$ and let $R = \{(1, 2), (2, 1)\}$. Then R is clearly irreflexive. In this case $R^2 = \{(1, 1), (2, 2)\}$, which is not irreflexive.

9.3

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$
b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
d) $\{(1, 3), (3, 1)\}$

1. In each case we use a 3×3 matrix, putting a 1 in position (i, j) if the pair (i, j) is in the relation and a 0 in position (i, j) if the pair (i, j) is not in the relation. For instance, in part **(a)** there are 1's in the first row, since each of the pairs $(1,1)$, $(1,2)$, and $(1,3)$ are in the relation, and there are 0's elsewhere.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

---

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

2. In each case we use a 4×4 matrix, putting a 1 in position (i, j) if the pair (i, j) is in the relation and a 0 in position (i, j) if the pair (i, j) is not in the relation.

a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. List the ordered pairs in the relations on {1, 2, 3} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

3. a) Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,2)^{\text{th}}$ entry is a 0, $(1,2)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,3)$, $(2,2)$, $(3,1)$, and $(3,3)$.
b) $(1,2)$, $(2,2)$, and $(3,2)$ **c)** $(1,1)$, (1.2), $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(3,3)$

4. List the ordered pairs in the relations on {1, 2, 3, 4} corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ **1**

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

4. a) Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,3)^{\text{th}}$ entry is a 0, $(1,3)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,3)$, and $(4,4)$.
b) $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(3,3)$, $(3,4)$, $(4,1)$, and $(1,4)$
c) $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,4)$, $(4,1)$, and $(4,3)$

---

5. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

5. An irreflexive relation (see the preamble to Exercise 11 in Section 9.1) is one in which no element is related to itself. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i). Equivalently, the relation is irreflexive if and only if every entry on the main diagonal of the matrix is 0.

6. How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

6. An asymmetric relation (see the preamble to Exercise 18 in Section 9.1) is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i), and there is no pair of 1's symmetrically placed around the main diagonal (i.e., we cannot have $m_{ij} = m_{ji} = 1$ for any values of i and j).

---

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers have if R is

a) $\{(a, b) \mid a > b\}$? **b)** $\{(a, b) \mid a \neq b\}$?
c) $\{(a, b) \mid a = b + 1\}$? **d)** $\{(a, b) \mid a = 1\}$?
e) $\{(a, b) \mid ab = 1\}$?

9. Note that the total number of entries in the matrix is $100^2 = 10{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 100, namely in position (a,b) where $a > b$. Thus the answer is the number of subsets of size 2 from a set of 100 elements, i.e., $C(100,2) = 4950$.

b) There is a 1 in the matrix at each position except the 100 positions on the main diagonal. Therefore the answer is $100^2 - 100 = 9900$.

c) There is a 1 in the matrix at each entry just below the main diagonal (i.e., in positions $(2,1)$, $(3,2)$, ..., $(100,99)$. Therefore the answer is 99.

d) The entire first row of this matrix corresponds to $a = 1$. Therefore the matrix has 100 nonzero entries.

e) This relation has only the one element $(1,1)$ in it, so the matrix has just one nonzero entry.

---

10. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 1000\}$ consisting of the first 1000 positive integers have if R is

a) $\{(a, b) \mid a \leq b\}$?
b) $\{(a, b) \mid a = b \pm 1\}$?
c) $\{(a, b) \mid a + b = 1000\}$?
d) $\{(a, b) \mid a + b \leq 1001\}$?
e) $\{(a, b) \mid a \neq 0\}$?

10. Note that the total number of entries in the matrix is $1000^2 = 1{,}000{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 1000, namely in position (a,b) where $a \leq b$, as well as 1's along the diagonal. Thus the answer is the number of subsets of size 2 from a set of 1000 elements, plus 1000, i.e., $C(1000,2) + 1000 = 499500 + 1000 = 500{,}500$.

b) There two 1's in each row of the matrix except the first and last rows, in which there is one 1. Therefore the answer is $998 \cdot 2 + 2 = 1998$.

c) There is a 1 in the matrix at each entry just above and to the left of the "anti-diagonal" (i.e., in positions $(1,999)$, $(2,998)$, ..., $(999,1)$. Therefore the answer is 999.

d) There is a 1 in the matrix at each entry on or above (to the left of) the "anti-diagonal." This is the same number of 1's as in part (a), so the answer is again 500,500.

e) The condition is trivially true (since $1 \leq a \leq 1000$), so all 1,000,000 entries are 1.

11. How can the matrix for $\overline{R}$, the complement of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

11. Since the relation $\overline{R}$ is the relation that contains the pair (a, b) (where a and b are elements of the appropriate sets) if and only if R does not contain that pair, we can form the matrix for $\overline{R}$ simply by changing all the 1's to 0's and 0's to 1's in the matrix for R.

---

12. How can the matrix for R^{-1}, the inverse of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

12. We take the transpose of the matrix, since we want the $(i, j)^{\text{th}}$ entry of the matrix for R^{-1} to be 1 if and only if the $(j, i)^{\text{th}}$ entry of R is 1.

---

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

13. Exercise 12 tells us how to do part **(a)** (we take the transpose of the given matrix $\mathbf{M}_R$, which in this case happens to be the matrix itself). Exercise 11 tells us how to do part **(b)** (we change 1's to 0's and 0's to 1's in $\mathbf{M}_R$). For part **(c)** we take the Boolean product of $\mathbf{M}_R$ with itself.

a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

---

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.
d) $R_1 \circ R_1$. **e)** $R_1 \oplus R_2$.

14. a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

c) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

d) The matrix is the Boolean product $\mathbf{M}_{R_1} \odot \mathbf{M}_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

---

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

15. We compute the Boolean powers of $\mathbf{M}_R$; thus $\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R$, $\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \mathbf{M}_R \odot \mathbf{M}_R^{[2]}$, and $\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \mathbf{M}_R \odot \mathbf{M}_R^{[3]}$.

a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

---

16. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1}, the inverse of R?

16. Since the matrix for R^{-1} is just the transpose of the matrix for R (see Exercise 12), the entries are the same collection of 0's and 1's, so there are k nonzero entries in $\mathbf{M}_{R^{-1}}$ as well.

---

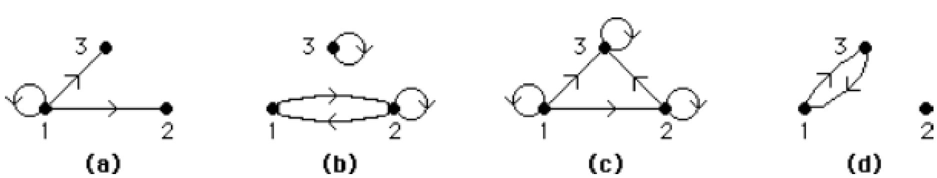
17. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{\overline{R}}$, the matrix representing $\overline{R}$, the complement of R?

17. The matrix for the complement has a 1 wherever the matrix for the relation has a 0, and vice versa. Therefore the number of nonzero entries in $\mathbf{M}_{\overline{R}}$ is $n^2 - k$, since these matrices have n rows and n columns.

---

18. Draw the directed graphs representing each of the relations from Exercise 1.

18. We draw the directed graphs, in each case with the vertex set being $\{1, 2, 3\}$ and an edge from i to j whenever (i, j) is in the relation.



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19. Draw the directed graphs representing each of the relations from Exercise 2.

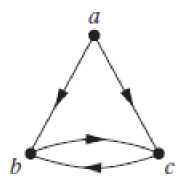
19. In each case we need a vertex for each of the elements, and we put in a directed edge from x to y if there is a 1 in position (x, y) of the matrix. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.

---

22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

23.



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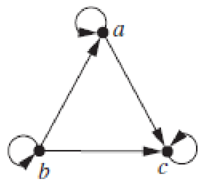

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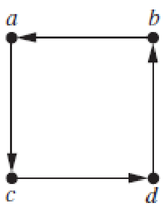

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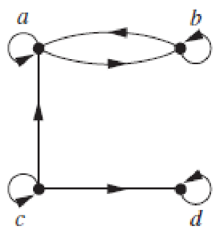


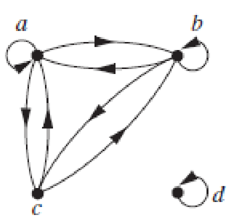
23. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,b),(a,c),(b,c),(c,b)\}$.

25. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,c),(b,a),(c,d),(d,b)\}$.

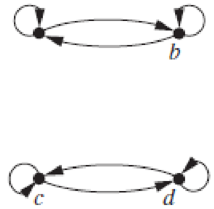
24. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,c)\}$.

26. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(c,d),(d,d)\}$.

27.



28.



27. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(d,d)\}$.

28. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$.

28. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$.

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30. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?

30. Clearly R is irreflexive if and only if there are no loops in the directed graph for R.

9.4

1. Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 2)$, and $(3, 0)$. Find the

a) reflexive closure of R. **b)** symmetric closure of R.

1. a) The reflexive closure of R is R together with all the pairs (a, a). Thus in this case the closure of R is $\{(0,0),(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(3,3)\}$.
b) The symmetric closure of R is R together with all the pairs (b, a) for which (a, b) is in R. For example, since $(1,2)$ is in R, we need to add $(2,1)$. Thus the closure of R is $\{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,0)\}$.

---

2. Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

2. When we add all the pairs (x, x) to the given relation we have all of $\mathbf{Z} \times \mathbf{Z}$; in other words, we have the relation that always holds.

---

3. Let R be the relation $\{(a, b) \mid a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

3. To form the symmetric closure we need to add all the pairs (b, a) such that (a, b) is in R. In this case, that means that we need to include pairs (b, a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a, b) such that b divides a. Thus the closure is $\{\,(a,b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\,\}$.

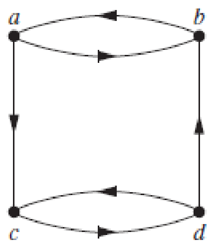
---

4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

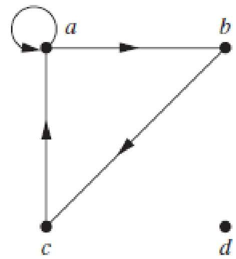
4. To form the reflexive closure, we simply need to add a loop at each vertex that does not already have one.

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.

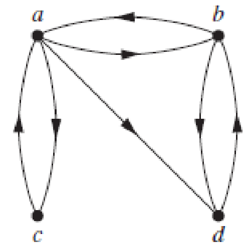
5.



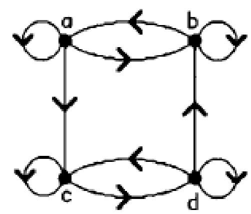
6.



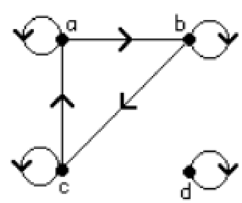
7.



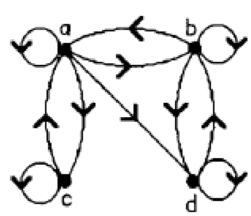
5. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



6. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



7. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



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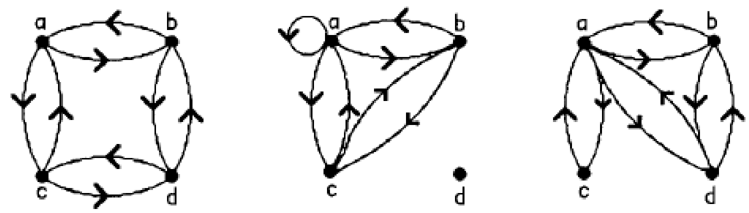
8. How can the directed graph representing the symmetric closure of a relation on a finite set be constructed from the directed graph for this relation?

8. To form the digraph of the symmetric closure, we simply need to add an edge from x to y whenever this edge is not already in the directed graph but the edge from y to x is.

9. Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.

9. We form the symmetric closure by taking the given directed graph and appending an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there); in other words, we append the edge (b, a) whenever we see the edge (a, b). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



(a) (b) (c)

---

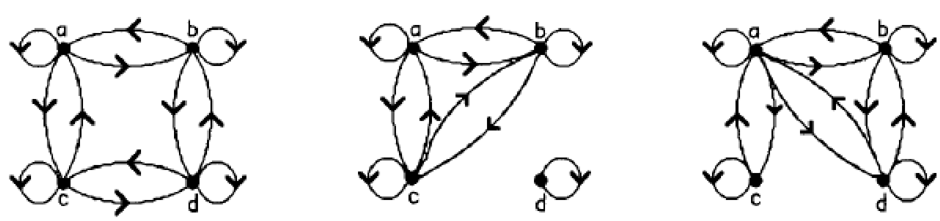
10. Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.

10. The symmetric closure was found in Example 2 to be the "is not equal to" relation. If we now make this relation reflexive as well, we will have the relation that always holds.

---

11. Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.

11. We are asked for the symmetric and reflexive closure of the given relation. We form it by taking the given directed graph and appending (1) a loop at each vertex at which there is not already a loop and (2) an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



(a) (b) (c)

12. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}_n$.

12. $\mathbf{M}_R \vee \mathbf{I}_n$ is by definition the same as $\mathbf{M}_R$ except that it has all 1's on the main diagonal. This must represent the reflexive closure of R, since this closure is the same as R except for the addition of all the pairs (x, x) that were not already present.

---

13. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^t$.

13. The symmetric closure of R is $R \cup R^{-1}$. The matrix for R^{-1} is $\mathbf{M}_R^t$, as we saw in Exercise 12 in Section 9.3. The matrix for the union of two relations is the join of the matrices for the two relations, as we saw in Section 9.3. Therefore the matrix representing the symmetric closure of R is indeed $\mathbf{M}_R \vee \mathbf{M}_R^t$.

---

14. Show that the closure of a relation R with respect to a property **P**, if it exists, is the intersection of all the relations with property **P** that contain R.

14. Suppose that the closure C exists. We must show that C is the intersection I of all the relations S that have property **P** and contain R. Certainly $I \subseteq C$, since C is one of the sets in the intersection. Conversely, by definition of closure, C is a subset of every relation S that has property **P** and contains R; therefore C is contained in their intersection.

---

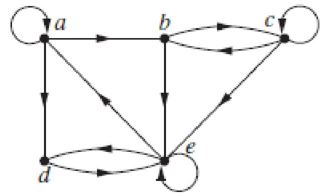
15. When is it possible to define the "irreflexive closure" of a relation R, that is, a relation that contains R, is irreflexive, and is contained in every irreflexive relation that contains R?

15. If R is already irreflexive, then it is clearly its own irreflexive closure. On the other hand if R is not irreflexive, then there is no relation containing R that is irreflexive, since the loop or loops in R prevent any such relation from being irreflexive. Thus in this case R has no irreflexive closure. This exercise shows essentially that the concept of "irreflexive closure" is rather useless, since no relation has one unless it is already irreflexive (in which case it is its own "irreflexive closure").

---

16. Determine whether these sequences of vertices are paths in this directed graph.

a) a, b, c, e
b) b, e, c, b, e
c) a, a, b, e, d, e
d) b, c, e, d, a, a, b
e) b, c, c, b, e, d, e, d
f) $a, a, b, b, c, c, b, e, d$



16. In each case, the sequence is a path if and only if there is an edge from each vertex in the sequence to the vertex following it.

a) This is a path. b) This is not a path (there is no edge from e to c). c) This is a path.

d) This is not a path (there is no edge from d to a). e) This is a path.

f) This is not a path (there is no loop at b).

---

17. Find all circuits of length three in the directed graph in Exercise 16.

17. A circuit of length 3 can be written as a sequence of 4 vertices, each joined to the next by an edge of the given directed graph, ending at the same vertex at which it began. There are several such circuits here, and we just have to be careful and systematically list them all. There are the circuits formed entirely by the loops: $aaaa$, $cccc$, and $eeee$. The triangles $abea$ and $adea$ also qualify. Two circuits start at b: $bccb$ and $beab$. There are two more circuits starting at c, namely $ccbc$ and $cbcc$. Similarly there are the circuits $deed$, $eede$ and $edee$, as well as the other trips around the triangle: $eabe$, $dead$, and $eade$.

---

18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.

a) a, b	**b)** b, a	**c)** b, b
d) a, e	**e)** b, d	**f)** c, d
g) d, d	**h)** e, a	**i)** e, c

18. In the language of Chapter 10, this digraph is strongly connected, so there will be a path from every vertex to every other vertex.

a) One path is a, b. b) One path is b, e, a. c) One path is b, c, b; a shorter one is just b.

d) One path is a, b, e. e) One path is b, e, d. f) One path is c, e, d.

g) One path is d, e, d. Another is the path of length 0 from d to itself.

h) One path is e, a. Another is $e, a, b, e, a, b, e, a, b, e, a$. i) One path is e, a, b, c.

19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3)$, $(2, 4)$, $(3, 1)$, $(3, 5)$, $(4, 3)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4.
d) R^5. **e)** R^6. **f)** R^*.

19. The way to form these powers is first to form the matrix representing R, namely

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

and then take successive Boolean powers of it to get the matrices representing R^2, R^3, and so on. Finally, for part **(f)** we take the join of the matrices representing R, R^2, $\ldots$, R^5. Since the matrix is a perfectly good way to express the relation, we will not list the ordered pairs.

a) The matrix for R^2 is the Boolean product of the matrix displayed above with itself, namely

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

b) The matrix for R^3 is the Boolean product of the first matrix displayed above with the answer to part **(a)**, namely

$$\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

c) The matrix for R^4 is the Boolean product of the first matrix displayed above with the answer to part **(b)**, namely

$$\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

d) The matrix for R^5 is the Boolean product of the first matrix displayed above with the answer to part **(c)**, namely

$$\mathbf{M}_{R^5} = \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

e) The matrix for R^6 is the Boolean product of the first matrix displayed above with the answer to part **(d)**, namely

$$\mathbf{M}_{R^6} = \mathbf{M}_R^{[6]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

f) The matrix for R^* is the join of the first matrix displayed above and the answers to parts **(a)** through **(d)**, namely

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \mathbf{M}_R^{[4]} \vee \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct non-stop airline flight from a to b. When is (a, b) in

a) R^2? b) R^3? c) R^*?

20. a) The pair (a, b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b—in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.

b) The pair (a, b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b—in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.

c) The pair (a, b) is in R^* precisely when it is possible to fly from a to b.

---

21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2? b) R^3? c) R^*?

21. **a)** The pair (a, b) is in R^2 if there is a person c other than a or b who is in a class with a and a class with b. Note that it is almost certain that (a, a) is in R^2, since as long as a is taking a class that has at least one other person in it, that person serves as the "c."

b) The pair (a, b) is in R^3 if there are persons c (different from a) and d (different from b and c) such that c is in a class with a, c is in a class with d, and d is in a class with b.

c) The pair (a, b) is in R^* if there is a sequence of persons, $c_0, c_1, c_2, \ldots, c_n$, with $n \geq 1$, such that $c_0 = a$, $c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is in at least one class with c_i.

---

22. Suppose that the relation R is reflexive. Show that R^* is reflexive.

22. Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$.

---

23. Suppose that the relation R is symmetric. Show that R^* is symmetric.

23. Suppose that $(a, b) \in R^*$; then there is a path from a to b in (the digraph for) R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R (following the given path backwards). This means that (b, a) is in R^* whenever (a, b) is, exactly what we needed to prove.

---

24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

24. It is certainly possibly for R^2 to contain some pairs (a, a). For example, let $R = \{(1,2), (2,1)\}$.

---

25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$

b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$

c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

25. Algorithm 1 finds the transitive closure by computing the successive powers and taking their join. We exhibit our answers in matrix form as $\mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \ldots \vee \mathbf{M}_R^{[n]} = \mathbf{M}_{R^*}$.

a)
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

---

26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.

a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$

b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$

c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$

d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$

26. a) We show the various matrices that are involved. First,

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{A}^{[2]} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \mathbf{A}^{[3]} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{A}.$$

It follows that $\mathbf{A}^{[4]} = \mathbf{A}^{[2]}$ and $\mathbf{A}^{[5]} = \mathbf{A}^{[3]}$. Therefore the answer $\mathbf{B}$, the meet of all the $\mathbf{A}$'s, is $\mathbf{A} \vee \mathbf{A}^{[2]}$, namely

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

b) For this and the remaining parts we just exhibit the matrices that arise.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \mathbf{A}^{[5]} \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

c)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d)

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{A}^{[2]} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \mathbf{B}$$

27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.

27. In Warshall's algorithm (Algorithm 2 in this section), we compute a sequence of matrices $\mathbf{W}_0$ (the matrix representing R), $\mathbf{W}_1$, $\mathbf{W}_2$, ..., $\mathbf{W}_n$, the last of which represents the transitive closure of R. Each matrix $\mathbf{W}_k$ comes from the matrix $\mathbf{W}_{k-1}$ in the following way. The $(i,j)^{\text{th}}$ entry of $\mathbf{W}_k$ is the "$\vee$" of the $(i,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$ with the "$\wedge$" of the $(i,k)^{\text{th}}$ entry and the $(k,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$. We will exhibit our solution by listing the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$, in that order; $\mathbf{W}_4$ represents the answer. In each case $\mathbf{W}_0$ is the matrix of the given relation. To compute the next matrix in the solution, we need to compute it one entry at a time, using the equation just discussed (the "$\vee$" of the corresponding entry in the previous matrix with the "$\wedge$" of two entries in the old matrix), i.e., as i and j each go from 1 to 4, we need to write down the $(i,j)^{\text{th}}$ entry using this formula. Note that in computing $\mathbf{W}_k$ the k^{th} row and the k^{th} column are unchanged, but some of the entries in other rows and columns may change.

a) $\begin{bmatrix} 0&1&0&0\\ 1&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{bmatrix}$ $\begin{bmatrix} 0&1&0&0\\ 1&1&1&0\\ 0&0&0&1\\ 1&1&0&0 \end{bmatrix}$ $\begin{bmatrix} 1&1&1&0\\ 1&1&1&0\\ 0&0&0&1\\ 1&1&1&0 \end{bmatrix}$ $\begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 0&0&0&1\\ 1&1&1&1 \end{bmatrix}$ $\begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$

b) $\begin{bmatrix} 0&0&0&0\\ 1&0&1&0\\ 1&0&0&1\\ 1&0&1&0 \end{bmatrix}$ $\begin{bmatrix} 0&0&0&0\\ 1&0&1&0\\ 1&0&0&1\\ 1&0&1&0 \end{bmatrix}$ $\begin{bmatrix} 0&0&0&0\\ 1&0&1&0\\ 1&0&0&1\\ 1&0&1&0 \end{bmatrix}$ $\begin{bmatrix} 0&0&0&0\\ 1&0&1&1\\ 1&0&0&1\\ 1&0&1&1 \end{bmatrix}$ $\begin{bmatrix} 0&0&0&0\\ 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1 \end{bmatrix}$

c) $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$ $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$ $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$ $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$ $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$

Note that the relation was already transitive, so each matrix in the sequence was the same.

d) $\begin{bmatrix} 1&0&0&1\\ 1&0&1&0\\ 1&1&0&1\\ 0&1&0&0 \end{bmatrix}$ $\begin{bmatrix} 1&0&0&1\\ 1&0&1&1\\ 1&1&0&1\\ 0&1&0&0 \end{bmatrix}$ $\begin{bmatrix} 1&0&0&1\\ 1&0&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$ $\begin{bmatrix} 1&0&0&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$ $\begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$

28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.

28. We compute the matrices $\mathbf{W}_i$ for $i = 0, 1, 2, 3, 4, 5$, and then $\mathbf{W}_5$ is the answer.

a)

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} = \mathbf{W}_5$$

b)

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \mathbf{W}_1 \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \mathbf{W}_3 = \mathbf{W}_4$$

$$\mathbf{W}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

c)

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{W}_3$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

d)

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \mathbf{W}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

29. Find the smallest relation containing the relation $\{(1,2),(1,4),(3,3),(4,1)\}$ that is

a) reflexive and transitive.

b) symmetric and transitive.

c) reflexive, symmetric, and transitive.

29. a) We need to include at least the transitive closure, which we can compute by Algorithm 1 or Algorithm 2 to be (in matrix form) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. All we need in addition is the pair $(2,2)$ in order to make the relation reflexive. Note that the result is still transitive (the addition of a pair (a,a) cannot make a transitive relation no longer transitive), so our answer is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

---

30. Finish the proof of the case when $a \neq b$ in Lemma 1.

30. Let m be the length of the shortest path from a to b, and let $a = x_0, x_1, \ldots, x_{m-1}, x_m = b$ be such a path. If $m > n-1$, then $m \geq n$, so $m+1 \geq n+1$, which means that not all of the vertices x_0, x_1, x_2, $\ldots$, x_m are distinct. Thus $x_i = x_j$ for some i and j with $0 \leq i < j \leq m$ (but not both $i = 0$ and $j = m$, since $a \neq b$). We can then excise the circuit from x_i to x_j, leaving a shorter path from a to b, namely $x_0, \ldots, x_i, x_{j+1}, \ldots, x_m$. This contradicts the choice of m. Therefore $m \leq n-1$, as desired.

---

34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.

34. All we need to do is make sure that all the pairs (x,x) are included. An easy way to accomplish this is to add them at the end, by setting $\mathbf{W} := \mathbf{W} \vee \mathbf{I}_n$.

9.5

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0,0), (1,1), (2,2), (3,3)\}$
b) $\{(0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3)\}$
c) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$
d) $\{(0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
e) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

1. In each case we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair $(1,1)$ is missing. It is also not transitive, since the pairs $(0,2)$ and $(2,3)$ are there, but not $(0,3)$.

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs $(1,3)$ and $(3,2)$ are present, but not $(1,2)$.

e) This relation would be an equivalence relation were the pair $(2,1)$ present. As it is, its absence makes the relation neither symmetric nor transitive.

---

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$
b) $\{(a, b) \mid a$ and b have the same parents$\}$
c) $\{(a, b) \mid a$ and b share a common parent$\}$
d) $\{(a, b) \mid a$ and b have met$\}$
e) $\{(a, b) \mid a$ and b speak a common language$\}$

2. a) This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).

b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).

c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

d) This is not an equivalence relation since it is clearly not transitive.

e) Again, just as in part (c), this is not transitive.

---

3. Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1 \text{ for all } x \in \mathbf{Z}\}$
d) $\{(f, g) \mid \text{ for some } C \in \mathbf{Z}, \text{ for all } x \in \mathbf{Z}, f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$

3. As in Exercise 1, we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation, one of the general form that two things are considered equivalent if they have the same "something" (see Exercise 9 for a formalization of this idea). In this case the "something" is the value at 1.

b) This is not an equivalence relation because it is not transitive. Let $f(x) = 0$, $g(x) = x$, and $h(x) = 1$ for all $x \in \mathbf{Z}$. Then f is related to g since $f(0) = g(0)$, and g is related to h since $g(1) = h(1)$, but f is not related to h since they have no values in common. By inspection we see that this relation is reflexive and symmetric.

c) This relation has none of the three properties. It is not reflexive, since $f(x) - f(x) = 0 \neq 1$. It is not symmetric, since if $f(x) - g(x) = 1$, then $g(x) - f(x) = -1 \neq 1$. It is not transitive, since if $f(x) - g(x) = 1$ and $g(x) - h(x) = 1$, then $f(x) - h(x) = 2 \neq 1$.

d) This is an equivalence relation. Two functions are related here if they differ by a constant. It is clearly reflexive (the constant is 0). It is symmetric, since if $f(x) - g(x) = C$, then $g(x) - f(x) = -C$. It is transitive, since if $f(x) - g(x) = C_1$ and $g(x) - h(x) = C_2$, then $f(x) - h(x) = C_3$, where $C_3 = C_1 + C_2$ (add the first two equations).

e) This relation is not reflexive, since there are lots of functions f (for instance, $f(x) = x$) that do not have the property that $f(0) = f(1)$. It is symmetric by inspection (the roles of f and g are the same). It is not transitive. For instance, let $f(0) = g(1) = h(0) = 7$, and let $f(1) = g(0) = h(1) = 3$; fill in the remaining values arbitrarily. Then f and g are related, as are g and h, but f is not related to h since $7 \neq 3$.

---

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

4. One relation is that a and b are related if they were born in the same U.S. state (with "not in a state of the U.S." counting as one state). Here the equivalence classes are the nonempty sets of students from each state. Another example is for a to be related to b if a and b have lived the same number of complete decades. The equivalence classes are the set of all 10-to-19 year-olds, the set of all 20-to-29 year-olds, and so on (the sets among these that are nonempty, that is). A third example is for a to be related to b if 10 is a divisor of the difference between a's age and b's age, where "age" means the whole number of years since birth, as of the first day of class. For each $i = 0, 1, \ldots, 9$, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit i.

---

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

5. Obviously there are many possible answers here. We can say that two buildings are equivalent if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year (as long as there was at least one building opened that year). For another example, we can define two buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on (one class for each n for which there is at least one n-story building). In our third example, partition the set of all buildings into two classes—those in which you do have a class this semester and those in which you don't. (We assume that each of these is nonempty.) Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class (including itself).

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6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

6. One way to partition the classes would be by level. At many schools, classes have three-digit numbers, the first digit of which is approximately the level of the course, so that courses numbered 100–199 are taken by freshman, 200–299 by sophomores, and so on. Formally, two classes are related if their numbers have the same digit in the hundreds column; the equivalence classes are the set of all 100-level classes, the set of all 200-level classes, and so on. A second example would focus on department. Two classes are equivalent if they are offered by the same department; for example, MATH 154 is equivalent to MATH 372, but not to EGR 141. The equivalence classes are the sets of classes offered by each department (the set of math classes, the set of engineering classes, and so on). A third—and more egocentric—classification would be to have one equivalence class be the set of classes that you have completed successfully and the other equivalence class to be all the other classes. Formally, two classes are equivalent if they have the same answer to the question, "Have I completed this class successfully?"

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7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of **F** and of **T**?

7. Two propositions are equivalent if their truth tables are identical. This relation is reflexive, since the truth table of a proposition is identical to itself. It is symmetric, since if p and q have the same truth table, then q and p have the same truth table. There is one technical point about transitivity that should be noted. We need to assume that the truth tables, as we consider them for three propositions p, q, and r, have the same

atomic variables in them. If we make this assumption (and it cannot hurt to do so, since adding information about extra variables that do not appear in a pair of propositions does not change the truth value of the propositions), then we argue in the usual way: if p and q have identical truth tables, and if q and r have identical truth tables, then p and r have that same common truth table. The proposition $\mathbf{T}$ is always true; therefore the equivalence class for this proposition consists of all propositions that are always true, no matter what truth values the atomic variables have. Recall that we call such a proposition a tautology. Therefore the equivalence class of $\mathbf{T}$ is the set of all tautologies. Similarly, the equivalence class of $\mathbf{F}$ is the set of all contradictions.

---

8. Let R be the relation on the set of all sets of real numbers such that $S\,R\,T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\mathbf{Z}$?

8. Recall (Definition 1 in Section 2.5) that two sets have the same cardinality if there is a bijection (one-to-one and onto function) from one set to the other. We must show that R is reflexive, symmetric, and transitive. Every set has the same cardinality as itself because of the identity function. If f is a bijection from S to T, then f^{-1} is a bijection from T to S, so R is symmetric. Finally, if f is a bijection from S to T and g is a bijection from T to U, then $g \circ f$ is a bijection from T to U, so R is transitive (see Exercise 33 in Section 2.3).

The equivalence class of $\{1, 2, 3\}$ is the set of all three-element sets of real numbers, including such sets as $\{4, 25, 1948\}$ and $\{e, \pi, \sqrt{2}\}$. Similarly, $[\mathbf{Z}]$ is the set of all infinite countable sets of real numbers (see Section 2.5), such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1, 2, 3\}$ (it's too small) or the set of all real numbers (it's too big). See Section 2.5 for more on countable sets.

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9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on A.

b) What are the equivalence classes of R?

9. This is an important exercise, since very many equivalence relations are of this form. (In fact, all of them are—see Exercise 10. A relation defined by a condition of the form "x and y are equivalent if and only if they have the same ..." is an equivalence relation. The function f here tells what about x and y are "the same.")

a) This relation is reflexive, since obviously $f(x) = f(x)$ for all $x \in A$. It is symmetric, since if $f(x) = f(y)$, then $f(y) = f(x)$ (this is one of the fundamental properties of equality). It is transitive, since if $f(x) = f(y)$ and $f(y) = f(z)$, then $f(x) = f(z)$ (this is another fundamental property of equality).

b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$. This is by definition just the inverse image of $f(x)$. Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

---

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

---

11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

11. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to its first 3 bits.

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12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

12. This follows from Exercise 9, where f is the function that takes a bit string of length $n \geq 3$ to its last $n - 3$ bits.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

13. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to the ordered pair (b_1, b_3), where b_1 is the first bit of the string and b_3 is the third bit of the string. Two bit strings agree on their first and third bits if and only if the corresponding ordered pairs for these two strings are equal ordered pairs.

---

14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n, the nth characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.

14. This follows from Exercise 9, where f is the function that takes a string of uppercase and lowercase English letters and changes all the lower case letters to their uppercase equivalents (and leaves the uppercase letters unchanged).

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15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

15. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x - y$. Therefore by Exercise 9 this is an equivalence relation. If we want a more explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a + b = b + a$. For symmetry, $((a,b),(c,d)) \in R$ if and only if $a + d = b + c$, which is equivalent to $c + b = d + a$, which is true if and only if $((c,d),(a,b)) \in R$. For transitivity, suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Thus we have $a + d = b + c$ and $c + e = d + f$. Adding, we obtain $a + d + c + e = b + c + d + f$. Simplifying, we have $a + e = b + f$, which tells us that $((a,b),(e,f)) \in R$.

---

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

16. This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a, b) to a/b, since clearly $ad = bc$ if and only if $a/b = c/d$.

If we want an explicit proof, we can argue as follows. For reflexivity, $((a, b), (a, b)) \in R$ because $a \cdot b = b \cdot a$. If $((a, b), (c, d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c, d), (a, b)) \in R$; this tells us that R is symmetric. Finally, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a, b), (e, f)) \in R$; this tells us that R is transitive.

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17. (*Requires calculus*)

a) Show that the relation R on the set of all differentiable functions from **R** to **R** consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.

b) Which functions are in the same equivalence class as the function $f(x) = x^2$?

17. a) This follows from Exercise 9, where the function f from the set of differentiable functions (from **R** to **R**) to the set of functions (from **R** to **R**) is the differentiation operator—i.e., f of a function g is the function g'. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) We are asking for all functions that have the same derivative that the function $f(x) = x^2$ has, i.e., all functions of x whose derivative is $2x$. In other words, we are asking for the general antiderivative of $2x$, and we know that $\int 2x = x^2 + C$, where C is any constant. Therefore the functions in the same equivalence class as $f(x) = x^2$ are all the functions of the form $g(x) = x^2 + C$ for some constant C. Indefinite integrals in calculus, then, give equivalence classes of functions as answers, not just functions.

18. (*Requires calculus*)

a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the nth derivative of $f(x)$.]

b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

18. a) This follows from Exercise 9, where the function f from the set of polynomials to the set of polynomials is the operator that takes the derivative n times—i.e., f of a function g is the function $g^{(n)}$. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "n^{th} derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) The third derivative of x^4 is $24x$. Since the third derivative of a polynomial of degree 2 or less is 0, the polynomials of the form $x^4 + ax^2 + bx + c$ have the same third derivative. Thus these are the functions in the same equivalence class as f.

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19. Let R be the relation on the set of all URLs (or Web addresses) such that $x\ R\ y$ if and only if the Web page at x is the same as the Web page at y. Show that R is an equivalence relation.

19. This follows from Exercise 9, where the function f from the set of all URLs to the set of all Web pages is the function that assigns to each URL the Web page for that URL.

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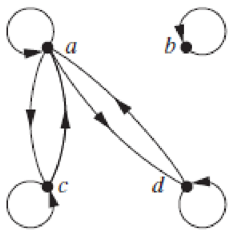
20. Let R be the relation on the set of all people who have visited a particular Web page such that $x\ R\ y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

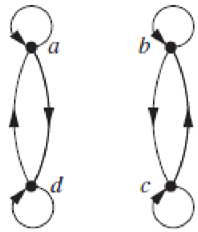
20. This follows from Exercise 9, where the function f from the set of people to the set of Web-traversing behaviors starting at the given particular Web page takes the person to the behavior that person exhibited.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

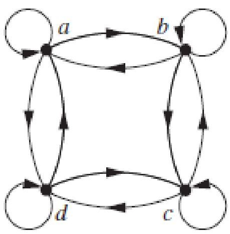
21.



22.



23.



21. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is not transitive, since the edges (c, d) and (d, c) are missing.

23. As in Exercise 21, this relation is not transitive, since several required edges are missing (such as (a, c)).

22. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is an equivalence relation, satisfying all three properties. The equivalence classes are $\{a, d\}$ and $\{b, c\}$.

---

24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

24. a) This is not an equivalence relation, since it is not symmetric.

b) This is an equivalence relation; one equivalence class consists of the first and third elements, and the other consists of the second and fourth elements.

c) This is an equivalence relation; one equivalence class consists of the first, second, and third elements, and the other consists of the fourth element.

25. Show that the relation R on the set of all bit strings such that $s\,R\,t$ if and only if s and t contain the same number of 1s is an equivalence relation.

25. This follows from Exercise 9, with f being the function from bit strings to nonnegative integers given by $f(s) =$ the number of 1's in s.

---

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 **b)** 1011 **c)** 11111 **d)** 01010101

30. **a)** all the strings whose first three bits are 010 **b)** all the strings whose first three bits are 101 **c)** all the strings whose first three bits are 111 **d)** all the strings whose first three bits are 010

---

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? **b)** 3? **c)** 6? **d)** −3?

35. We have by definition that $[n]_5 = \{\, i \mid i \equiv n \pmod 5 \,\}$.

a) $[2]_5 = \{\, i \mid i \equiv 2 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

b) $[3]_5 = \{\, i \mid i \equiv 3 \pmod 5 \,\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

c) $[6]_5 = \{\, i \mid i \equiv 6 \pmod 5 \,\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

d) $[-3]_5 = \{\, i \mid i \equiv -3 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ (the same as $[2]_5$)

---

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo m.

a) $\{\, 4 + 2n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 0, 2, 4, \ldots\}$ **b)** $\{\, 4 + 3n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 1, 4, 7, \ldots\}$

c) $\{\, 4 + 6n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 4, 10, 16, \ldots\}$ **d)** $\{\, 4 + 8n \mid n \in \mathbf{Z} \,\} = \{\ldots, -4, 4, 12, 20, \ldots\}$

---

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) *No* **b)** *Yes* **c)** *Help*

38. In each case we need to allow all strings that agree with the given string if we ignore the case in which the letters occur.

a) $\{NO, No, nO, no\}$

b) $\{YES, YEs, YeS, Yes, yES, yEs, yeS, yes\}$

c) $\{HELP, HELp, HElP, HElp, HeLP, HeLp, HelP, Help, hELP, hELp, hElP, hElp, heLP, heLp, helP, help\}$

---

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$ **b)** $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

41. The sets in a partition must be nonempty, pairwise disjoint, and have as their union all of the underlying set.

a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

b) This is a partition. **c)** This is a partition.

d) This is not a partition, since none of the sets includes the element 3.

---

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$

b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$

c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$

d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

42. a) This is a partition, since it satisfies the definition.

b) This is not a partition, since the subsets are not disjoint.

c) This is a partition, since it satisfies the definition.

d) This is not a partition, since the union of the subsets leaves out 0.

---

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain $3k$ ones for some nonnegative integer k; the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k; and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

43. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set of 256 bit strings of length 8.

a) This is a partition, since strings must begin either 1 or 0, and those that begin 0 must continue with either 0 or 1 in their second position. It is clear that the three subsets satisfy the conditions.

b) This is not a partition, since these subsets are not pairwise disjoint. The string 00000001, for example, contains both 00 and 01.

c) This is clearly a partition. Each of these four subsets contains 64 bit strings, and no two of them overlap.

d) This is not a partition, because the union of these subsets is not the entire set. For example, the string 00000010 is in none of the subsets.

e) This is a partition. Each bit string contains some number of 1's. This number can be identified in exactly one way as of the form $3k$, the form $3k+1$, or the form $3k+2$, where k is a nonnegative integer; it really is just looking at the equivalence classes of the number of 1's modulo 3.

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers

b) the set of positive integers and the set of negative integers

c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

d) the set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

44. a) This is clearly a partition. b) This is not a partition, since 0 is in neither set.

c) This is a partition by the division algorithm.

d) This is a partition, since the second set mentioned is the set of all number between -100 and 100, inclusive.

e) The first two sets are not disjoint (4 is in both), so this is not a partition.

45. Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

a) the set of pairs (x, y), where x or y is odd; the set of pairs (x, y), where x is even; and the set of pairs (x, y), where y is even

b) the set of pairs (x, y), where both x and y are odd; the set of pairs (x, y), where exactly one of x and y is odd; and the set of pairs (x, y), where both x and y are even

c) the set of pairs (x, y), where x is positive; the set of pairs (x, y), where y is positive; and the set of pairs (x, y), where both x and y are negative

d) the set of pairs (x, y), where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y), where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y), where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y), where $3 \nmid x$ and $3 \nmid y$

e) the set of pairs (x, y), where $x > 0$ and $y > 0$; the set of pairs (x, y), where $x > 0$ and $y \leq 0$; the set of pairs (x, y), where $x \leq 0$ and $y > 0$; and the set of pairs (x, y), where $x \leq 0$ and $y \leq 0$

f) the set of pairs (x, y), where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y), where $x = 0$ and $y \neq 0$; and the set of pairs (x, y), where $x \neq 0$ and $y = 0$

45. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set $\mathbf{Z} \times \mathbf{Z}$.

a) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2, 3)$, for example, is in both of the first two subsets listed.

b) This is a partition. Every pair satisfies exactly one of the conditions listed about the parity of x and y, and clearly these subsets are nonempty.

c) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2, 3)$, for example, is in both of the first two subsets listed. Also, $(0, 0)$ is in none of the subsets.

d) This is a partition. Every pair satisfies exactly one of the conditions listed about the divisibility of x and y by 3, and clearly these subsets are nonempty.

e) This is a partition. Every pair satisfies exactly one of the conditions listed about the positiveness of x and y, and clearly these subsets are nonempty.

f) This is not a partition, because the union of these subsets is not all of $\mathbf{Z} \times \mathbf{Z}$. In particular, $(0, 0)$ is in none of the parts.

46. Which of these are partitions of the set of real numbers?

a) the negative real numbers, $\{0\}$, the positive real numbers

b) the set of irrational numbers, the set of rational numbers

c) the set of intervals $[k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

d) the set of intervals $(k, k+1)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

e) the set of intervals $(k, k+1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

f) the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$

46. a) This is a partition, since it satisfies the definition.

b) This is a partition, since it satisfies the definition.

c) This is not a partition, since the intervals are not disjoint (they share endpoints).

d) This is not a partition, since the union of the subsets leaves out the integers.

e) This is a partition, since it satisfies the definition.

f) This is a partition, since it satisfies the definition. Each equivalence class consists of all real numbers with a fixed fractional part.

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47. List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

47. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$

b) $\{(0,0), (0,1), (1,0), (1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$

c) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$

d) $\{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)\}$

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48. List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.

a) $\{a, b\}, \{c, d\}, \{e, f, g\}$

b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$

c) $\{a, b, c, d\}, \{e, f, g\}$

d) $\{a, c, e, g\}, \{b, d\}, \{f\}$

A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

48. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

b) $\{(a,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(f,e),(f,f),(g,g)\}$

c) $\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),$
$(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

d) $\{(a,a),(a,c),(a,e),(a,g),(c,a),(c,c),(c,e),(c,g),(e,a),(e,c),(e,e),(e,g),(g,a),(g,c),(g,e),(g,g),$
$(b,b),(b,d),(d,b),(d,d),(f,f)\}$

9.6

1. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (1, 1), (2, 2), (3, 3)}

b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)}

d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)}

1. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) Clearly this relation is reflexive because each of 0, 1, 2, and 3 is related to itself. The relation is also antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c$ so a is related to c (because the relation is reflexive). This is just the equality relation on $\{0, 1, 2, 3\}$; more generally, the equality relation on any set satisfies all three conditions and is therefore a partial ordering. (It is the smallest partial ordering; reflexivity insures that every partial ordering contains at least all the pairs (a, a).)

b) This is not a partial ordering, because although the relation is reflexive, it is not antisymmetric (we have $2\,R\,3$ and $3\,R\,2$, but $2 \neq 3$), and not transitive ($3\,R\,2$ and $2\,R\,0$, but 3 is not related to 0).

c) This is a partial ordering, because it is clearly reflexive; is antisymmetric (we just need to note that $(1, 2)$ is the only pair in the relation with unequal components); and is transitive (for the same reason).

d) This is a partial ordering because it is the "less than or equal to" relation on $\{1, 2, 3\}$ together with the isolated point 0.

e) This is not a partial ordering. The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

2. Which of these relations on {0, 1, 2, 3} are partial orderings? Determine the properties of a partial ordering that the others lack.

a) {(0, 0), (2, 2), (3, 3)}

b) {(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)}

c) {(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)}

d) {(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)}

e) {(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)}

2. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) This relation is not reflexive because 1 is not related to itself. Therefore R is not a partial ordering. The relation is antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c \neq 1$ so a is related to c.

b) This is a partial ordering, because it is reflexive and the pairs $(2,0)$ and $(2,3)$ will not introduce any violations of antisymmetry or transitivity.

c) This is not a partial ordering, because it is not transitive: $3\,R\,1$ and $1\,R\,2$, but 3 is not related to 2. It is reflexive and the pairs $(1,2)$ and $(3,1)$ will not introduce any violations of antisymmetry.

d) This is not a partial ordering, because it is not transitive: $1\,R\,2$ and $2\,R\,0$, but 1 is not related to 0. It is reflexive and the nonreflexive pairs will not introduce any violations of antisymmetry.

e) The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

---

3. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) $a = b$ or a is an ancestor of b?

d) a and b have a common friend?

3. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since nobody is taller than himself, this relation is not reflexive so (S, R) cannot be a poset.

b) To be not taller means to be exactly the same height or shorter. Two different people x and y could have the same height, in which case $x\,R\,y$ and $y\,R\,x$ but $x \neq y$, so R is not antisymmetric and this is not a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is an ancestor of b, then b cannot be an ancestor of a (for one thing, an ancestor needs to be born before any descendant), so the relation is vacuously antisymmetric. If a is an ancestor of b, and b is an ancestor of c, then by the way "ancestor" is defined, we know that a is an ancestor of b; thus R is transitive.

d) This relation is not antisymmetric. Let a and b be any two distinct friends of yours. Then $a\,R\,b$ and $b\,R\,a$, but $a \neq b$.

4. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is no shorter than b?

b) a weighs more than b?

c) $a = b$ or a is a descendant of b?

d) a and b do not have a common friend?

4. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S, R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S, R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b, then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b, and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

---

5. Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

5. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial partial ordering. (It is the smallest partial partial ordering; reflexivity insures that every partial order contains at least all the pairs (a, a).)

b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

c) This is a poset, as explained in Example 1.

d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that $2 \nmid 2$. (It also is not antisymmetric and not transitive.)

---

6. Which of these are posets?

a) $(\mathbf{R}, =)$ **b)** $(\mathbf{R}, <)$ **c)** $(\mathbf{R}, \leq)$ **d)** $(\mathbf{R}, \neq)$

6. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs (a, a).)

b) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs (a, a).

c) This is a poset, very similar to Example 1.

d) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

---

7. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

7. a) This relation is $\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,3)\}$. It is not antisymmetric because $(1,2)$ and $(2,1)$ are both in the relation, but $1 \neq 2$. We can see this visually by the pair of 1's symmetrically placed around the main diagonal at positions $(1,2)$ and $(2,1)$. Therefore this matrix does not represent a partial order.

b) This matrix represents a partial order. Reflexivity is clear. The only other pairs in the relation are $(1,2)$ and $(1,3)$, and clearly neither can be part of a counterexample to antisymmetry or transitivity.

c) A little trial and error shows that this relation is not transitive ($(4,1)$ and $(1,3)$ are present, but not $(4,3)$) and therefore not a partial order.

---

8. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

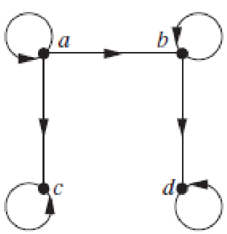
8. a) This relation is $\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$. It is clearly reflexive and antisymmetric. The only pairs that might present problems with transitivity are the nondiagonal pairs, $(2,1)$ and $(1,3)$. If the relation were to be transitive, then we would also need the pair $(2,3)$ in the relation. Since it is not there, the relation is not a partial order.

b) Reasoning as in part (a), we see that this relation is a partial order, since the pair $(3,1)$ can cause no problem with transitivity.

c) A little trial and error shows that this relation is not transitive ($(1,3)$ and $(3,4)$ are present, but not $(1,4)$) and therefore not a partial order.

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

9.



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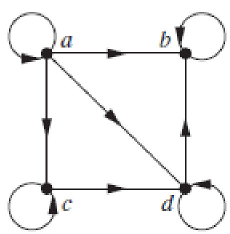


11.

9. This relation is not transitive (there are arrows from a to b and from b to d, but there is no arrow from a to d), so it is not a partial order.

10. This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

11. This relation is a partial order, since it has all three properties—it is reflexive (there is an arrow at each point), antisymmetric (there are no pairs of arrows going in opposite directions between two different points), and transitive (there is no missing arrow from some x to some z when there were arrows from x to y and y to z).

---

12. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R. The poset (S, R^{-1}) is called the **dual** of (S, R).

12. This follows immediately from the definition. Clearly R^{-1} is reflexive if R is. For antisymmetry, suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, whence $(b,a) \notin R^{-1}$. Finally, if $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, so $(c,a) \in R$ (since R is transitive), and therefore $(a,c) \in R^{-1}$; thus R^{-1} is transitive.

13. Find the duals of these posets.

a) $(\{0, 1, 2\}, \leq)$ **b)** $(\mathbf{Z}, \geq)$
c) $(P(\mathbf{Z}), \supseteq)$ **d)** $(\mathbf{Z}^+, |)$

13. The dual of a poset is the poset with the same underlying set and with the relation defined by declaring a related to b if and only if $b \preceq a$ in the given poset.

a) The dual relation to $\leq$ is $\geq$, so the dual poset is $(\{0,1,2\}, \geq)$. Explicitly it is the set $\{(0,0), (1,0), (1,1), (2,0), (2,1), (2,2)\}$.

b) The dual relation to $\geq$ is $\leq$, so the dual poset is $(\mathbf{Z}, \leq)$.

c) The dual relation to $\supseteq$ is $\subseteq$, so the dual poset is $(P(\mathbf{Z}), \subseteq)$.

d) There is no symbol generally used for the "is a multiple of" relation, which is the dual to the "divides" relation in this part of the exercise. If we let R be the relation such that aRb if and only if $b \mid a$, then the answer can be written $(\mathbf{Z}^+, R)$.

---

14. Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?

a) 5, 15 **b)** 6, 9 **c)** 8, 16 **d)** 7, 7

14. **a)** These are comparable, since $5 \mid 15$.

b) These are not comparable since neither divides the other.

c) These are comparable, since $8 \mid 16$.

d) These are comparable, since $7 \mid 7$.

---

15. Find two incomparable elements in these posets.

a) $(P(\{0, 1, 2\}), \subseteq)$ **b)** $(\{1, 2, 4, 6, 8\}, |)$

15. We need to find elements such that the relation holds in neither direction between them. The answers we give are not the only ones possible.

a) One such pair is $\{1\}$ and $\{2\}$. These are both subsets of $\{0,1,2\}$, so they are in the poset, but neither is a subset of the other.

b) Neither 6 nor 8 divides the other, so they are incomparable.

16. Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual "less than" relation,

a) find all pairs in $S \times S$ less than $(2, 3)$.

b) find all pairs in $S \times S$ greater than $(3, 1)$.

c) draw the Hasse diagram of the poset $(S \times S, \preccurlyeq)$.

16. a) We need either a number less than 2 in the first coordinate, or a 2 in the first coordinate and a number less than 3 in the second coordinate. Therefore the answer is $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, and $(2,2)$.

b) We need either a number greater than 3 in the first coordinate, or a 3 in the first coordinate and a number greater than 1 in the second coordinate. Therefore the answer is $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(3,2)$, $(3,3)$, and $(3,4)$.

c) The Hasse diagram is a straight line with 16 points on it, since this is a total order. The pair $(4,4)$ is at the top, $(4,3)$ beneath it, $(4,2)$ beneath that, and so on, with $(1,1)$ at the bottom. To save space, we will not actually draw this picture.

---

17. Find the lexicographic ordering of these n-tuples:

a) $(1, 1, 2), (1, 2, 1)$ **b)** $(0, 1, 2, 3), (0, 1, 3, 2)$

c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$

17. We find the first coordinate (from left to right) at which the tuples differ and place first the tuple with the smaller value in that coordinate.

a) Since $1 = 1$ in the first coordinate, but $1 < 2$ in the second coordinate, $(1,1,2) < (1,2,1)$.

b) The first two coordinates agree, but $2 < 3$ in the third, so $(0,1,2,3) < (0,1,3,2)$.

c) Since $0 < 1$ in the first coordinate, $(0,1,1,1,0) < (1,0,1,0,1)$.

---

18. Find the lexicographic ordering of these strings of lowercase English letters:

a) *quack, quick, quicksilver, quicksand, quacking*

b) *open, opener, opera, operand, opened*

c) *zoo, zero, zoom, zoology, zoological*

18. a) The string *quack* comes first, since it is an initial substring of *quacking*, which comes next (since the other three strings all begin *qui*, not *qua*). Similarly, these last three strings are in the order *quick*, *quicksand*, *quicksilver*.

b) The order is *open*, *opened*, *opener*, *opera*, *operand*.

c) The order is *zero*, *zoo*, *zoological*, *zoology*, *zoom*.

19. Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.

19. All the strings that begin with 0 precede all those that begin with 1. The 0 comes first. Next comes 0001, which begins with three 0's, then 001, which begins with two 0's. Among the strings that begin 01, the order is $01 < 010 < 0101 < 011$. Putting this all together, we have $0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$.

---

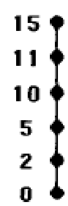
20. Draw the Hasse diagram for the "greater than or equal to" relation on $\{0, 1, 2, 3, 4, 5\}$.

20. The Hasse diagram for this total order is a straight line, as shown, with 0 at the top (it is the "largest" element under the "is greater than or equal to" relation) and 5 at the bottom.

---

21. Draw the Hasse diagram for the "less than or equal to" relation on $\{0, 2, 5, 10, 11, 15\}$.

21. This is a totally ordered set, so the Hasse diagram is linear.



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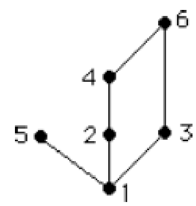
22. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6\}$. **b)** $\{3, 5, 7, 11, 13, 16, 17\}$.

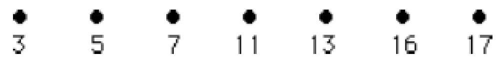
c) $\{2, 3, 5, 10, 11, 15, 25\}$. **d)** $\{1, 3, 9, 27, 81, 243\}$.

22. In each case we put a above b and draw a line between them if $b \mid a$ but there is no element c other than a and b such that $b \mid c$ and $c \mid a$.

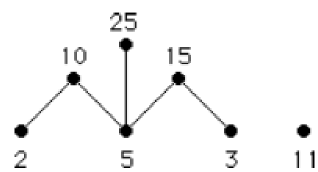
a) Note that 1 divides all numbers, so the numbers on the second level from the bottom are the primes.



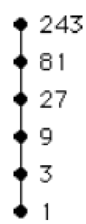
b) In this case these numbers are pairwise relatively prime, so there are no lines in the Hasse diagram.



c) Note that we can place the points as we wish, as long as a is above b when $b \mid a$.



d) In this case these numbers each divide the next, so the Hasse diagram is a straight line.



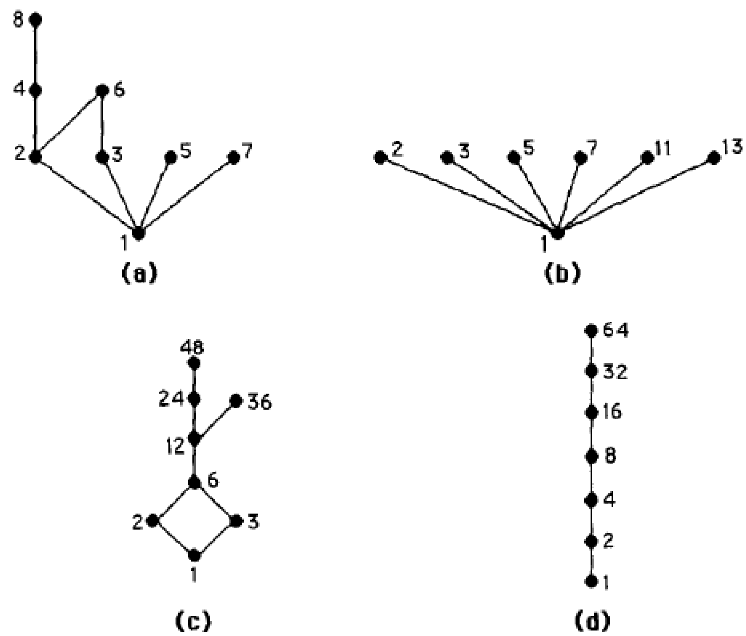
23. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. b) $\{1, 2, 3, 5, 7, 11, 13\}$.

c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.

d) $\{1, 2, 4, 8, 16, 32, 64\}$.

23. We put x above y if y divides x. We draw a line between x and y, where y divides x, if there is no number z in our set, other than x or y, such that $y \mid z \wedge z \mid x$. Note that in part **(b)** the numbers other than 1 are all (relatively) prime, so the Hasse diagram is short and wide, whereas in part **(d)** the numbers all divide one another, so the Hasse diagram is tall and narrow.



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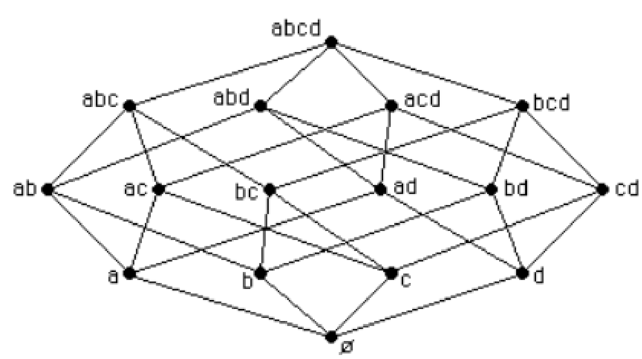
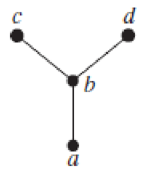
24. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$.

24. This picture is a four-dimensional cube. We draw the sets with k elements at level k: the empty set at level 0 (the bottom), the entire set at level 4 (the top).

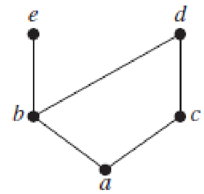


In Exercises 25–27 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

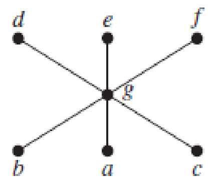
25.



26.



27.



d)
e)
f)
g)
h)
34. Ar
18
a)
b)
c)
d)
e)
f)

25. We need to include every pair (x, y) for which we can find a path going upward in the diagram from x to y. We also need to include all the reflexive pairs (x, x). Therefore the relation is the following set of pairs: $\{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,d)\}$.

26. The procedure is the same as in Exercise 25: $\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,d),(d,d),(e,e)\}$

27. The procedure is the same as in Exercise 25: $\{(a,a),(a,d),(a,e),(a,f),(a,g),(b,b),(b,d),(b,e),(b,f),(b,g),(c,c),(c,d),(c,e),(c,f),(c,g),(d,d),(e,e),(f,f),(g,d),(g,e),(g,f),(g,g)\}$.

28. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?

28. In this problem $a \preceq b$ when $a \mid b$. For (a,b) to be in the covering relation, we need a to be a proper divisor of b but we also must have no element in our set $\{1, 2, 3, 4, 6, 12\}$ being a proper multiple of a and a proper divisor of b. For example, $(2,12)$ is not in the covering relation, since $2 \mid 6$ and $6 \mid 12$. With this understanding it is easy to list the pairs in the covering relation: $(1,2)$, $(1,3)$, $(2,4)$, $(2,6)$, $(3,6)$, $(4,12)$,and $(6,12)$.

29. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of S, where $S = \{a, b, c\}$?

29. In this problem $X \preceq Y$ when $X \subseteq Y$. For (X, Y) to be in the covering relation, we need X to be a proper subset of Y but we also must have no subset strictly between X and Y. For example, $(\{a\}, \{a, b, c\})$ is not in the covering relation, since $\{a\} \subset \{a, b\}$ and $\{a, b\} \subset \{a, b, c\}$. With this understanding it is easy to list the pairs in the covering relation: $(\emptyset, \{a\})$, $(\emptyset, \{b\})$, $(\emptyset, \{c\})$, $(\{a\}, \{a, b\})$, $(\{a\}, \{a, c\})$, $(\{b\}, \{a, b\})$, $(\{b\}, \{b, c\})$, $(\{c\}, \{a, c\})$, $(\{c\}, \{b, c\})$, $(\{a, b\}, \{a, b, c\})$, $(\{a, c\}, \{a, b, c\})$, and $(\{b, c\}, \{a, b, c\})$.

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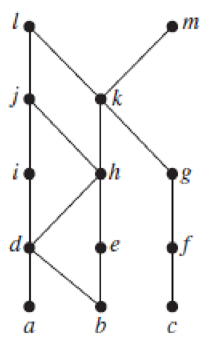
32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{a, b, c\}$.

f) Find the least upper bound of $\{a, b, c\}$, if it exists.

g) Find all lower bounds of $\{f, g, h\}$.

h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

32. **a)** The maximal elements are the ones with no other elements above them, namely l and m.

b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

c) There is no greatest element, since neither l nor m is greater than the other.

d) There is no least element, since neither a nor b is less than the other.

e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l, and m are the elements fitting this description.

f) Since k is less than both l and m, it is the least upper bound of a, b, and c.

g) No element is less than both f and h, so there are no lower bounds.

h) Since there are no lower bounds, there can be no greatest lower bound.

---

33. Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{3, 5\}$.

f) Find the least upper bound of $\{3, 5\}$, if it exists.

g) Find all lower bounds of $\{15, 45\}$.

h) Find the greatest lower bound of $\{15, 45\}$, if it exists.

33. It is helpful in this exercise to draw the Hasse diagram.

a) Maximal elements are those that do not divide any other elements of the set. In this case 24 and 45 are the only numbers that meet that requirement.

b) Minimal elements are those that are not divisible by any other elements of the set. In this case 3 and 5 are the only numbers that meet that requirement.

c) A greatest element would be one that all the other elements divide. The only two candidates (maximal elements) are 24 and 45, and since neither divides the other, we conclude that there is no greatest element.

d) A least element would be one that divides all the other elements. The only two candidates (minimal elements) are 3 and 5, and since neither divides the other, we conclude that there is no least element.

e) We want to find all elements that both 3 and 5 divide. Clearly only 15 and 45 meet this requirement.

f) The least upper bound is 15 since it divides 45 (see part **(e)**).

g) We want to find all elements that divide both 15 and 45. Clearly only 3, 5, and 15 meet this requirement.

h) The number 15 is the greatest lower bound, since both 3 and 5 divide it (see part **(g)**).

34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

e) Find all upper bounds of $\{2, 9\}$.

f) Find the least upper bound of $\{2, 9\}$, if it exists.

g) Find all lower bounds of $\{60, 72\}$.

h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

34. The reader should draw the Hasse diagram to aid in answering these questions.

a) Clearly the numbers 27, 48, 60, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.

b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.

c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

d) There is no least element, since there is no number in the set that divides both 2 and 9.

e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.

f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

g) We need to find numbers in the list that are divisors of both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.

h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

35. Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}, \subseteq)$.

a) Find the maximal elements.

b) Find the minimal elements.

c) Is there a greatest element?

d) Is there a least element?

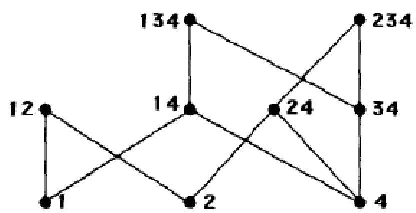
e) Find all upper bounds of $\{\{2\}, \{4\}\}$.

f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.

g) Find all lower bounds of $\{\{1,3,4\}, \{2,3,4\}\}$.

h) Find the greatest lower bound of $\{\{1,3,4\}, \{2,3,4\}\}$, if it exists.

35. To help us answer the questions, we will draw the Hasse diagram, with the commas and braces eliminated in the labels, for readability.



a) The maximal elements are the ones without any elements lying above them in the Hasse diagram, namely $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

b) The minimal elements are the ones without any elements lying below them in the Hasse diagram, namely $\{1\}$, $\{2\}$, and $\{4\}$.

c) There is no greatest element, since there is more than one maximal element, none of which is greater than the others.

d) There is no least element, since there is more than one minimal element, none of which is less than the others.

e) The upper bounds are the sets containing both $\{2\}$ and $\{4\}$ as subsets, i.e., the sets containing both 2 and 4 as elements. Pictorially, these are the elements lying above both $\{2\}$ and $\{4\}$ (in the sense of there being a path in the diagram), namely $\{2,4\}$ and $\{2,3,4\}$.

f) The least upper bound is an upper bound that is less than every other upper bound. We found the upper bounds in part **(e)**, and since $\{2,4\}$ is less than (i.e., a subset of) $\{2,3,4\}$, we conclude that $\{2,4\}$ is the least upper bound.

g) To be a lower bound of both $\{1,3,4\}$ and $\{2,3,4\}$, a set must be a subset of each, and so must be a subset of their intersection, $\{3,4\}$. There are only two such subsets in our poset, namely $\{3,4\}$ and $\{4\}$. In the diagram, these are the points which lie below (in the path sense) both $\{1,3,4\}$ and $\{2,3,4\}$.

h) The greatest lower bound is a lower bound that is greater than every other lower bound. We found the lower bounds in part **(g)**, and since $\{3,4\}$ is greater than (i.e., a superset of) $\{4\}$, we conclude that $\{3,4\}$ is the greatest lower bound.

36. Give a poset that has

a) a minimal element but no maximal element.

b) a maximal element but no minimal element.

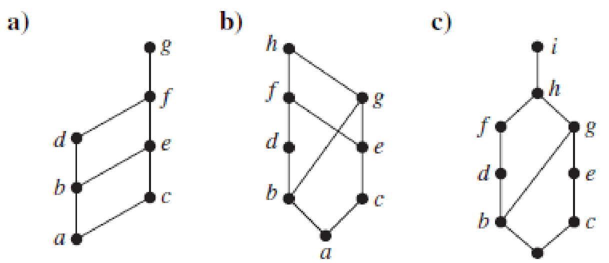
c) neither a maximal nor a minimal element.

36. a) One example is the natural numbers under "is less than or equal to." Here 1 is the (only) minimal element, and there are no maximal elements.

b) Dual to part (a), the answer is the natural numbers under "is greater than or equal to."

c) Combining the answers for the first two parts, we look at the set of integers under "is less than or equal to." Clearly there are no maximal or minimal elements.

43. Determine whether the posets with these Hasse diagrams are lattices.



43. In each case, we need to check whether every pair of elements has both a least upper bound and a greatest lower bound.

a) This is a lattice. If we want to find the l.u.b. or g.l.b. of two elements in the same vertical column of the Hasse diagram, then we simply take the higher or lower (respectively) element. If the elements are in different columns, then to find the g.l.b. we follow the diagonal line upward from the element on the left, and then continue upward on the right, if necessary to reach the element on the right. For example, the l.u.b. of d and c is f; and the l.u.b. of a and e is e. Finding greatest lower bounds in this poset is similar.

b) This is not a lattice. Elements b and c have f, g, and h as upper bounds, but none of them is a l.u.b.

c) This is a lattice. By considering all the pairs of elements, we can verify that every pair of them has a l.u.b. and a g.l.b. For example, b and e have g and a filling these roles, respectively.

44. Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$ **b)** $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

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47. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs (A, C). Here A is an authority level, which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3). A category C is a subset of the set of all projects {*Cheetah, Impala, Puma*}. (Names of animals are often used as code names for projects in companies.)

a) Is information permitted to flow from (*Proprietary*, {*Cheetah, Puma*}) into (*Restricted*, {*Puma*})?

b) Is information permitted to flow from (*Restricted*, {*Cheetah*}) into (*Registered*, {*Cheetah, Impala*})?

c) Into which classes is information from (*Proprietary*, {*Cheetah, Puma*}) permitted to flow?

d) From which classes is information permitted to flow into the security class (*Restricted*, {*Impala, Puma*})?

47. The needed definitions are in Example 25.

a) No. The authority level of the first pair (1) is less than or equal to (less than, in this case) that of the second (2); but the subset of the first pair is not a subset of that of the second.

b) Yes. The authority level of the first pair (2) is less than or equal to (less than, in this case) that of the second (3); and the subset of the first pair is a subset of that of the second.

c) The classes into which information can flow are those classes whose authority level is at least as high as *Proprietary*, and whose subset is a superset of {*Cheetah, Puma*}. We can list these classes: (*Proprietary*, {*Cheetah, Puma*}), (*Restricted*, {*Cheetah, Puma*}), (*Registered*, {*Cheetah, Puma*}), (*Proprietary*, {*Cheetah, Puma, Impala*}), (*Restricted*, {*Cheetah, Puma, Impala*}), and (*Registered*, {*Cheetah, Puma, Impala*}).

d) The classes from which information can flow are those classes whose authority level is at least as low as *Restricted*, and whose subset is a subset of {*Impala, Puma*}, namely (*Nonproprietary*,{*Impala,Puma*}), (*Proprietary*,{*Impala,Puma*}), (*Restricted*,{*Impala,Puma*}), (*Nonproprietary*,{*Impala*}), (*Proprietary*, {*Impala*}), (*Restricted*,{*Impala*}), (*Nonproprietary*,{*Puma*}), (*Proprietary*,{*Puma*}), (*Restricted*,{*Puma*}), (*Nonproprietary*,∅), (*Proprietary*,∅), and (*Restricted*,∅).

50. Show that every totally ordered set is a lattice.

51. Show that every finite lattice has a least element and a greatest element.

50. This issue was already dealt with in our solution to Exercise 44, parts **(b)** and **(c)**. If $(S, \leq)$ is a total (linear) order, then the least upper bound of two elements is the larger one, and their greatest lower bound is the smaller.

51. This follows immediately from Exercise 45. To be more specific, according to Exercise 45, there is a least upper bound (respectively, a greatest lower bound) for the entire finite lattice. This element is by definition a greatest element (respectively, a least element).

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52. Give an example of an infinite lattice with

a) neither a least nor a greatest element.

b) a least but not a greatest element.

c) a greatest but not a least element.

d) both a least and a greatest element.

52. By Exercise 50, we can try to choose our examples from among total orders, such as subsets of $\mathbf{Z}$ under $\leq$.
a) $(\mathbf{Z}, \leq)$ b) $(\mathbf{Z}^+, \leq)$ c) $(\mathbf{Z}^-, \leq)$, where $\mathbf{Z}^-$ is the set of negative integers d) $(\{1\}, \leq)$

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62. Find a compatible total order for the divisibility relation on the set $\{1, 2, 3, 6, 8, 12, 24, 36\}$.

62. Since a larger number can never divide a smaller one, the "is less than or equal to" relation on any set is a compatible total order for the divisibility relation. This gives $1 \prec_t 2 \prec_t 3 \prec_t 6 \prec_t 8 \prec_t 12 \prec_t 24 \prec_t 36$.

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63. Find all compatible total orderings for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$ from Example 26.

63. Clearly 1 must come first, and 20 must follow each element except possibly 12. The relative positions of 2, 4, and 12 are fixed. The 5 can go anywhere, as long as it lies between 1 and 20. Following these guidelines, we see that the following seven total orderings are the ones compatible with the given relation: $1 \prec 5 \prec 2 \prec 4 \prec 12 \prec 20$, $1 \prec 2 \prec 5 \prec 4 \prec 12 \prec 20$, $1 \prec 2 \prec 4 \prec 5 \prec 12 \prec 20$, $1 \prec 2 \prec 4 \prec 12 \prec 5 \prec 20$, $1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$, $1 \prec 2 \prec 5 \prec 4 \prec 20 \prec 12$, $1 \prec 2 \prec 4 \prec 5 \prec 20 \prec 12$.
