10.1

The examples and exercises give a good picture of the ways in which graphs can model various real world applications. In constructing graph models you need to determine what the vertices will represent, what the edges will represent, whether the edges will be directed or undirected, whether loops should be allowed, and whether a simple graph or multigraph is more appropriate.

---

1. Draw graph models, stating the type of graph (from Table 1) used, to represent airline routes where every day there are four flights from Boston to Newark, two flights from Newark to Boston, three flights from Newark to Miami, two flights from Miami to Newark, one flight from Newark to Detroit, two flights from Detroit to Newark, three flights from Newark to Washington, two flights from Washington to Newark, and one flight from Washington to Miami, with

a) an edge between vertices representing cities that have a flight between them (in either direction).

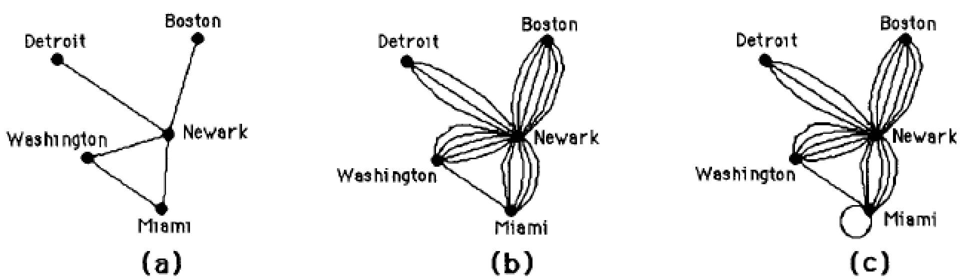
b) an edge between vertices representing cities for each flight that operates between them (in either direction).

c) an edge between vertices representing cities for each flight that operates between them (in either direction), plus a loop for a special sightseeing trip that takes off and lands in Miami.

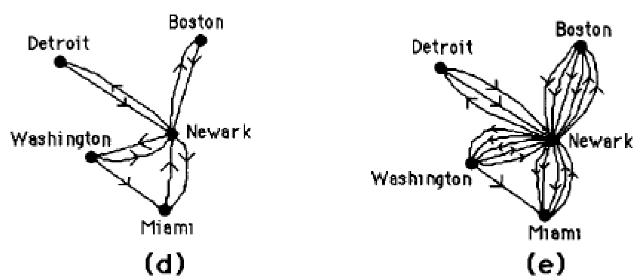
d) an edge from a vertex representing a city where a flight starts to the vertex representing the city where it ends.

e) an edge for each flight from a vertex representing a city where the flight begins to the vertex representing the city where the flight ends.

1. In part **(a)** we have a simple graph, with undirected edges, no loops or multiple edges. In part **(b)** we have a multigraph, since there are multiple edges (making the figure somewhat less than ideal visually). In part **(c)** we have the same picture as in part **(b)** except that there is now a loop at one vertex; thus this is a pseudograph.



In part **(d)** we have a directed graph, the directions of the edges telling the directions of the flights; note that the **antiparallel edges** (pairs of the form (u, v) and (v, u)) are not parallel. In part **(e)** we have a directed multigraph, since there are parallel edges.



---

2. What kind of graph (from Table 1) can be used to model a highway system between major cities where

a) there is an edge between the vertices representing cities if there is an interstate highway between them?

b) there is an edge between the vertices representing cities for each interstate highway between them?

c) there is an edge between the vertices representing cities for each interstate highway between them, and there is a loop at the vertex representing a city if there is an interstate highway that circles this city?

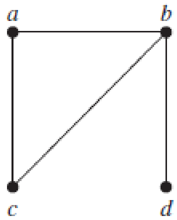
2. a) A simple graph would be the model here, since there are no parallel edges or loops, and the edges are undirected.

b) A multigraph would, in theory, be needed here, since there may be more than one interstate highway between the same pair of cities.

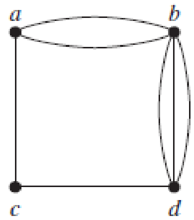
c) A pseudograph is needed here, to allow for loops.

For Exercises 3–9, determine whether the graph shown has directed or undirected edges, whether it has multiple edges, and whether it has one or more loops. Use your answers to determine the type of graph in Table 1 this graph is.

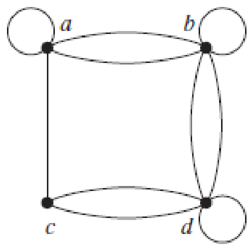
3.



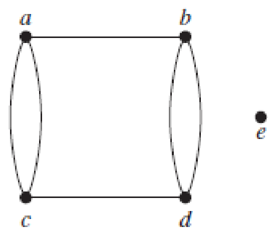
4.



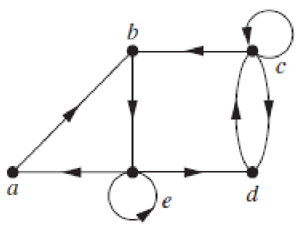
5.



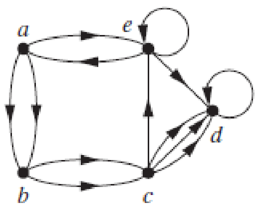
6.



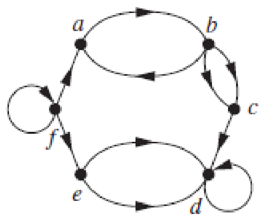
7.



8.



9.



3. This is a simple graph; the edges are undirected, and there are no parallel edges or loops.

4. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

5. This is a pseudograph; the edges are undirected, but there are loops and parallel edges.

6. This is a multigraph; the edges are undirected, and there are no loops, but there are parallel edges.

7. This is a directed graph; the edges are directed, but there are no parallel edges. (Loops and antiparallel edges—see the solution to Exercise 1d for a definition—are allowed in a directed graph.)

8. This is a directed multigraph; the edges are directed, and there are parallel edges.

9. This is a directed multigraph; the edges are directed, and there is a set of parallel edges.

10. For each undirected graph in Exercises 3–9 that is not simple, find a set of edges to remove to make it simple.

10. The graph in Exercise 3 is simple. The multigraph in Exercise 4 can be made simple by removing one of the edges between a and b, and two of the edges between b and d. The pseudograph in Exercise 5 can be made simple by removing the three loops and one edge in each of the three pairs of parallel edges. The multigraph in Exercise 6 can be made simple by removing one of the edges between a and c, and one of the edges between b and d. The other three are not undirected graphs. (Of course removing any supersets of the answers given here are equally valid answers; in particular, we could remove *all* the edges in each case.)

---

11. Let G be a simple graph. Show that the relation R on the set of vertices of G such that uRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

11. In a simple graph, edges are undirected. To show that R is symmetric we must show that if uRv, then vRu. If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. A simple graph does not allow loops; that is if there is an edge associated with $\{u, v\}$, then $u \neq v$. Thus uRu never holds, and so by definition R is irreflexive.

---

12. Let G be an undirected graph with a loop at every vertex. Show that the relation R on the set of vertices of G such that $u\,R\,v$ if and only if there is an edge associated to $\{u, v\}$ is a symmetric, reflexive relation on G.

12. If $u\,R\,v$, then there is an edge joining vertices u and v, and since the graph is undirected, this is also an edge joining vertices v and u. This means that $v\,R\,u$. Thus the relation is symmetric. The relation is reflexive because the loops guarantee that $u\,R\,u$ for each vertex u.

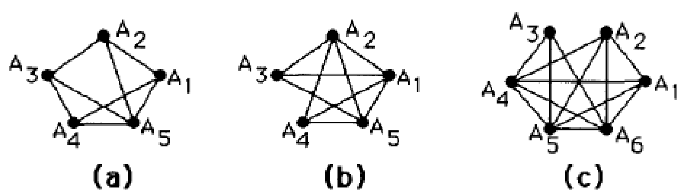
13. The **intersection graph** of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

a) $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$,
$A_3 = \{1, 3, 5, 7, 9\}$, $A_4 = \{5, 6, 7, 8, 9\}$,
$A_5 = \{0, 1, 8, 9\}$

b) $A_1 = \{\ldots, -4, -3, -2, -1, 0\}$,
$A_2 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$,
$A_3 = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$,
$A_4 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$,
$A_5 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

c) $A_1 = \{x \mid x < 0\}$,
$A_2 = \{x \mid -1 < x < 0\}$,
$A_3 = \{x \mid 0 < x < 1\}$,
$A_4 = \{x \mid -1 < x < 1\}$,
$A_5 = \{x \mid x > -1\}$,
$A_6 = \mathbf{R}$

13. In each case we draw a picture of the graph in question. All are simple graphs. An edge is drawn between two vertices if the sets for the two vertices have at least one element in common. For example, in part **(a)** there is an edge between vertices A_1 and A_2 because there is at least one element common to A_1 and A_2 (in fact there are three such elements). There is no edge between A_1 and A_3 since $A_1 \cap A_3 = \emptyset$.

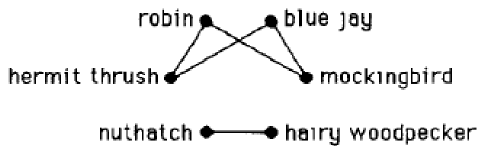


14. Use the niche overlap graph in Figure 11 to determine the species that compete with hawks.

14. Since there are edges from Hawk to Crow, Owl, and Raccoon, the graph is telling us that the hawk competes with these three animals.

15. Construct a niche overlap graph for six species of birds, where the hermit thrush competes with the robin and with the blue jay, the robin also competes with the mockingbird, the mockingbird also competes with the blue jay, and the nuthatch competes with the hairy woodpecker.

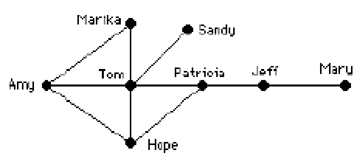
15. We draw a picture of the graph in question, which is a simple graph. Two vertices are joined by an edge if we are told that the species compete (such as robin and mockingbird) but there is no edge between pairs of species that are not given as competitors (such as robin and blue jay).



---

16. Draw the acquaintanceship graph that represents that Tom and Patricia, Tom and Hope, Tom and Sandy, Tom and Amy, Tom and Marika, Jeff and Patricia, Jeff and Mary, Patricia and Hope, Amy and Hope, and Amy and Marika know each other, but none of the other pairs of people listed know each other.

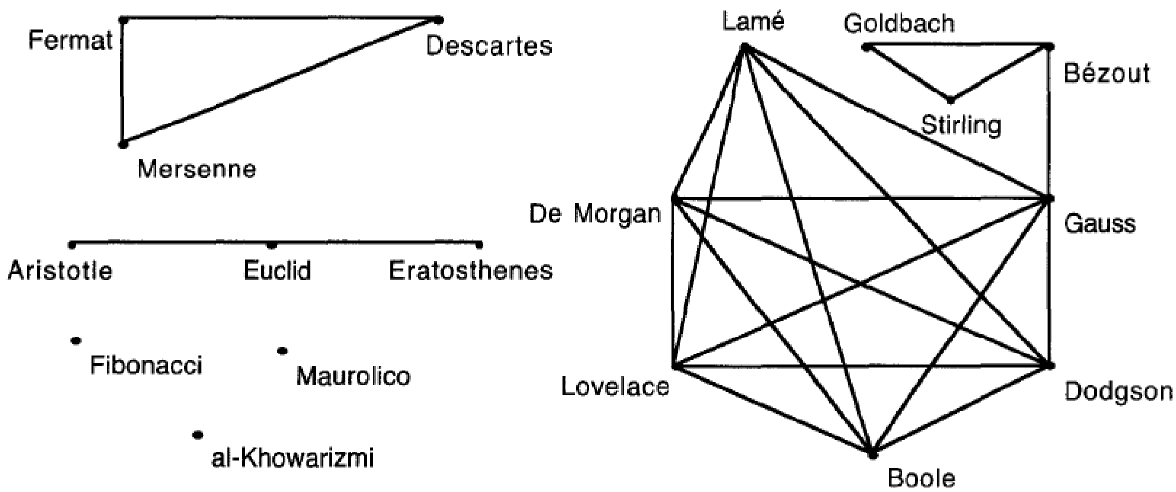
16. Each person is represented by a vertex, with an edge between two vertices if and only if the people are acquainted.



---

17. We can use a graph to represent whether two people were alive at the same time. Draw such a graph to represent whether each pair of the mathematicians and computer scientists with biographies in the first five chapters of this book who died before 1900 were contemporaneous. (Assume two people lived at the same time if they were alive during the same year.)

17. Here are the persons to be included, listed in order of birth year: Aristotle (384–322 B.C.E.), Euclid (325–265 B.C.E.), Eratosthenes (276–194 B.C.E.), al-Khowarizmi (780–850), Fibonacci (1170–1250), Maurolico (1494–1575), Mersenne (1588–1648), Descartes (1596–1650), Fermat (1601–1665), Goldbach (1690–1764), Stirling (1692–1770), Bézout (1730–1783), Gauss (1777–1855), Lamé (1795–1870), De Morgan (1806–1871), Lovelace (1815–1852), Boole (1815–1864), and Dodgson (1832–1898). We draw the graph by connecting two people if their date ranges overlap. Note that there is a complete subgraph (see Section 10.2) consisting of the last six people listed. A few of the vertices are isolated (again see Section 10.2). In all our graph has 17 vertices and 22 edges. A graph like this is called an **interval graph**, since each vertex can be associated with an interval of real numbers; it is a special case of an **intersection graph**, where two vertices are adjacent if the sets associated with those vertices have a nonempty intersection (see Exercise 13).



---

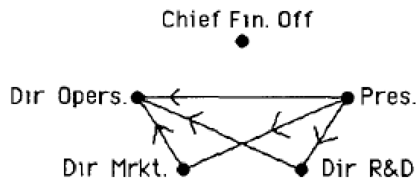
18. Who can influence Fred and whom can Fred influence in the influence graph in Example 2?

18. Fred influences Brian, since there is an edge from Fred to Brian. Yvonne and Deborah influence Fred, since there are edges from these vertices to Fred.

---

19. Construct an influence graph for the board members of a company if the President can influence the Director of Research and Development, the Director of Marketing, and the Director of Operations; the Director of Research and Development can influence the Director of Operations; the Director of Marketing can influence the Director of Operations; and no one can influence, or be influenced by, the Chief Financial Officer.

19. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u can influence v. For instance the Chief Financial Officer is an isolated vertex since she is influenced by no one and influences no one.



---

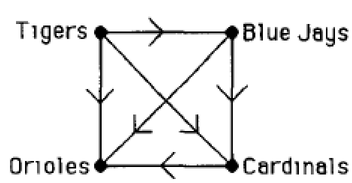
20. Which other teams did Team 4 beat and which teams beat Team 4 in the round-robin tournament represented by the graph in Figure 13?

20. Team four beat the vertices to which there are edges from Team four, namely only Team three. The other teams—Team one, Team two, Team five, and Team six—all beat Team four, since there are edges from them to Team four.

---

21. In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tigers beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles. Model this outcome with a directed graph.

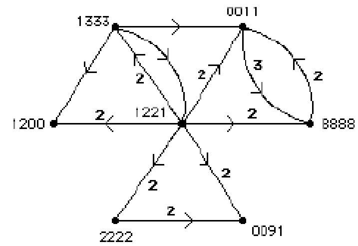
21. We draw a picture of the graph in question, which is a directed graph. We draw an edge from u to v if we are told that u beat v.



---

22. Construct the call graph for a set of seven telephone numbers 555-0011, 555-1221, 555-1333, 555-8888, 555-2222, 555-0091, and 555-1200 if there were three calls from 555-0011 to 555-8888 and two calls from 555-8888 to 555-0011, two calls from 555-2222 to 555-0091, two calls from 555-1221 to each of the other numbers, and one call from 555-1333 to each of 555-0011, 555-1221, and 555-1200.

22. This is a directed multigraph with one edge from a to b for each call made by a to b. Rather than draw the parallel edges with parallel lines, we have indicated what is intended by writing a numeral on the edge to indicate how many calls were made, if it was more than one.



---

23. Explain how the two telephone call graphs for calls made during the month of January and calls made during the month of February can be used to determine the new telephone numbers of people who have changed their telephone numbers.

23. We could compile a list of phone numbers (the labels on the vertices) in the February call graph that were not present in January, and a list of the January numbers missing in February. For each number in each list, we could make a list of the numbers they called or were called by, using the edges in the call graphs. Then we could look for February lists that were very similar to January lists. If we found a new February number that had almost the same calling pattern as a defunct January number, then we might suspect that these numbers belonged to the same person, who had recently changed his or her number.

---

24. **a)** Explain how graphs can be used to model electronic mail messages in a network. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

b) Describe a graph that models the electronic mail sent in a network in a particular week.

24. This is similar to the use of directed graphs to model telephone calls.

a) We can have a vertex for each mailbox or e-mail address in the network, with a directed edge between two vertices if a message is sent from the tail of the edge to the head.

b) As in part (a) we use a directed edge for each message sent during the week.

25. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?

25. For each e-mail address (the labels on the vertices), we could make a list of the other addresses they sent messages to or received messages from. If we see two addresses that had almost the same communication pattern, then we might suspect that these addresses belonged to the same person, who had recently changed his or her e-mail address.

---

26. How can a graph that models e-mail messages sent in a network be used to find electronic mail mailing lists used to send the same message to many different e-mail addresses?

26. Vertices with thousands or millions of edges going out from them could be the senders of such mass mailings. The collection of heads of these edges would be the mailing lists themselves.

---

27. Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

27. The vertices represent the people at the party. Because it is possible that a knows b's name but not vice versa, we need a directed graph. We will include an edge associated with (u, v) if and only if u knows v's name. There is no need for multiple edges (either a knows b's name or he doesn't). One could argue that we should not clutter the model with loops, because obviously everyone knows her own name. On the other hand, it certainly would not be wrong to include loops, especially if we took the instructions literally.

---

28. Describe a graph model that represents a subway system in a large city. Should edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

28. We make the subway stations the vertices, with an edge from station u to station v if there is a train going from u to v without stopping. It is quite possible that some segments are one-way, so we should use directed edges. (If there are no one-way segments, then we could use undirected edges.) There would be no need for multiple edges, unless we had two kinds of edges, maybe with different colors, to represent local and express trains. In that case, there could be parallel edges of different colors between the same vertices, because both a local and an express train might travel the same segment. There would be no point in having loops, because no passenger would want to travel from a station back to the same station without stopping.

29. For each course at a university, there may be one or more other courses that are its prerequisites. How can a graph be used to model these courses and which courses are prerequisites for which courses? Should edges be directed or undirected? Looking at the graph model, how can we find courses that do not have any prerequisites and how can we find courses that are not the prerequisite for any other courses?

29. We should use a directed graph, with the vertices being the courses and the edges showing the prerequisite relationship. Specifically, an edge from u to v means that course u is a prerequisite for course v. Courses that do not have any prerequisites are the courses with in-degree 0, and courses that are not the prerequisite for any other courses have out-degree 0. An interesting question would be how to model courses that are co-requisites (in two different senses—either courses u and v must be taken at the same time, or course u must be taken before course v or in the same semester as course v).

---

30. Describe a graph model that represents the positive recommendations of movie critics, using vertices to represent both these critics and all movies that are currently being shown.

30. A bipartite graph (this terminology is introduced in the next section) works well here. There are two types of vertices—one type representing the critics and one type representing the movies. There is an edge between vertex c (a critic vertex) and vertex m (a movie vertex) if and only if the critic represented by c has positively recommended the movie represented by m. There are no edges between critic vertices and there are no edges between movie vertices.

---

31. Describe a graph model that represents traditional marriages between men and women. Does this graph have any special properties?

31. For this to be interesting, we want the graph to model all marriages, not just ones that are currently active. (In the latter case, for the Western world, there would be at most one edge incident to each vertex.) So we let the set of vertices be a set of people (for example, all the people in North America who lived at any point in the 20th century), and two vertices are joined by an edge if the two people were ever married. Since laws in the 20th century allowed only marriages between persons of the opposite sex, and ignoring complications caused by sex-change operations, we note that this graph has the property that there are two types of vertices (men and women), and every edge joins vertices of opposite types. In the next section we learn that the word used to describe a graph like this is *bipartite*.

---

32. Which statements must be executed before S_6 is executed in the program in Example 8? (Use the precedence graph in Figure 10.)

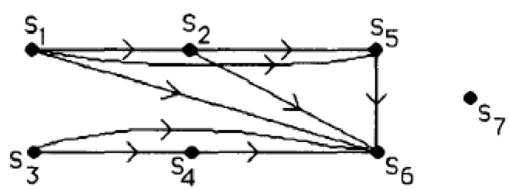
32. The model says that the statements for which there are edges to S_6 must be executed before S_6, namely the statements S_1, S_2, S_3, and S_4.

---

33. Construct a precedence graph for the following program:

S_1: $x := 0$
S_2: $x := x + 1$
S_3: $y := 2$
S_4: $z := y$
S_5: $x := x + 2$
S_6: $y := x + z$
S_7: $z := 4$

33. We draw a picture of the directed graph in question. There is an edge from u to v if the assignment made in u can possibly influence the assignment made in v. For example, there is an edge from S_3 to S_6, since the assignment in S_3 changes the value of y, which then influences the value of z (in S_4) and hence has a bearing on S_6. We assume that the statements are to be executed in the given order, so, for example, we do not draw an edge from S_5 to S_2.



---

34. Describe a discrete structure based on a graph that can be used to model airline routes and their flight times. [*Hint:* Add structure to a directed graph.]

34. The vertices in the directed graph represent cities. Whenever there is a nonstop flight from city A to city B, we put a directed edge into our directed graph from vertex A to vertex B, and furthermore we label that edge with the flight time. Let us see how to incorporate this into the mathematical definition. Let us call such a thing a directed graph with weighted edges. It is defined to be a triple (V, E, W), where (V, E) is a directed graph (i.e., V is a set of vertices and E is a set of ordered pairs of elements of V) and W is a function from E to the set of nonnegative real numbers. Here we are simply thinking of $W(e)$ as the weight of edge e, which in this case is the flight time.

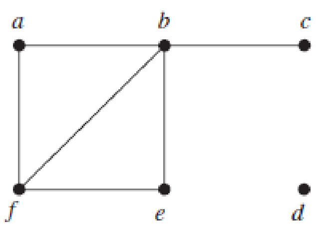
SECTION 10.2 Graph Terminology and Special Types of Graphs

Graph theory is sometimes jokingly called the "theory of definitions," because so many terms can be—and have been—defined for graphs. A few of the most important concepts are given in this section; others appear in the rest of this chapter and the next, in the exposition and in the exercises. As usual with definitions, it is important to understand exactly what they are saying. You should construct some examples for each definition you encounter—examples both of the thing being defined and of its absence. Some students find it useful to build a dictionary as they read, including their examples along with the formal definitions.

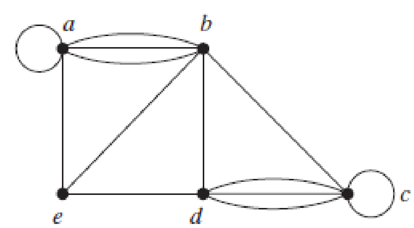
The handshaking theorem (that the sum of the degrees of the vertices in a graph equals twice the number of edges), although trivial to prove, is quite handy, as Exercise 55, for example, illustrates. Be sure to look at Exercise 43, which deals with the problem of when a sequence of numbers can possibly be the degrees of the vertices of a simple graph. Some interesting subtleties arise there, as you will discover when you try to draw the graphs. Many arguments in graph theory tend to be rather ad hoc, really getting down to the nitty gritty, and Exercise 43c is a good example. Exercise 51 is really a combinatorial problem; such problems abound in graph theory, and entire books have been written on counting graphs of various types. The notion of **complementary graph**, *introduced in Exercise 59, will appear again later in this chapter, so it would be wise to look at the exercises dealing with it.*

In Exercises 1–3 find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.

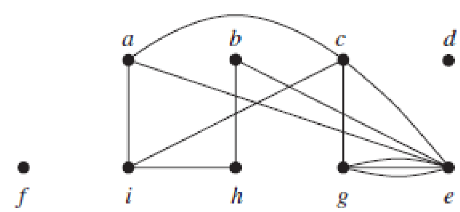
1.



2.



3.



1. There are 6 vertices here, and 6 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 1$ (and hence c is pendant), $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 2$, and $\deg(f) = 3$. Note that the sum of the degrees is $2 + 4 + 1 + 0 + 2 + 3 = 12$, which is twice the number of edges.

2. In this pseudograph there are 5 vertices and 13 edges. The degree of vertex a is 6, since in addition to the 4 nonloops incident to a, there is a loop contributing 2 to the degree. The degrees of the other vertices are $\deg(b) = 6$, $\deg(c) = 6$, $\deg(d) = 5$, and $\deg(e) = 3$. There are no pendant or isolated vertices in this pseudograph.

3. There are 9 vertices here, and 12 edges. The degree of each vertex is the number of edges incident to it. Thus $\deg(a) = 3$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 0$ (and hence d is isolated), $\deg(e) = 6$, $\deg(f) = 0$ (and hence f is isolated), $\deg(g) = 4$, $\deg(h) = 2$, and $\deg(i) = 3$. Note that the sum of the degrees is $3 + 2 + 4 + 0 + 6 + 0 + 4 + 2 + 3 = 24$, which is twice the number of edges.

---

4. Find the sum of the degrees of the vertices of each graph in Exercises 1–3 and verify that it equals twice the number of edges in the graph.

4. For the graph in Exercise 1, the sum is $2+4+1+0+2+3 = 12 = 2 \cdot 6$; there are 6 edges. For the pseudograph in Exercise 2, the sum is $6+6+6+5+3 = 26 = 2 \cdot 13$; there are 13 edges. For the pseudograph in Exercise 3, the sum is $3+2+4+0+6+0+4+2+3 = 24 = 2 \cdot 12$; there are 12 edges.

---

5. Can a simple graph exist with 15 vertices each of degree five?

5. By Theorem 2 the number of vertices of odd degree must be even. Hence there cannot be a graph with 15 vertices of odd degree 5. (We assume that the problem was meant to imply that the graph contained only these 15 vertices.)

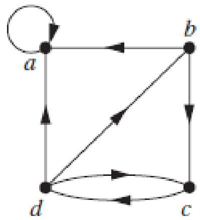
---

6. Show that the sum, over the set of people at a party, of the number of people a person has shaken hands with, is even. Assume that no one shakes his or her own hand.

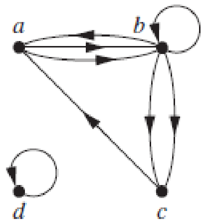
6. Model this problem by letting the vertices of a graph be the people at the party, with an edge between two people if they shake hands. Then the degree of each vertex is the number of people the person that vertex represents shakes hands with. By Theorem 1 the sum of the degrees is even (it is $2e$).

In Exercises 7–9 determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

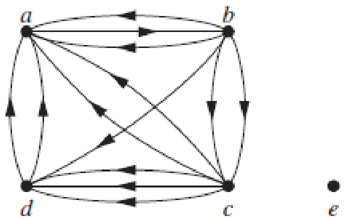
7.



8.



9.



7. This directed graph has 4 vertices and 7 edges. The in-degree of vertex a is $\deg^-(a) = 3$ since there are 3 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$ since only the loop has a as its initial vertex. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 2$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 3$. As a check we see that the sum of the in-degrees and the sum of the out-degrees are equal (both are equal to 7).

8. In this directed multigraph there are 4 vertices and 8 edges. The degrees are $\deg^-(a) = 2$, $\deg^+(a) = 2$, $\deg^-(b) = 3$, $\deg^+(b) = 4$, $\deg^-(c) = 2$, $\deg^+(c) = 1$, $\deg^-(d) = 1$, and $\deg^+(d) = 1$.

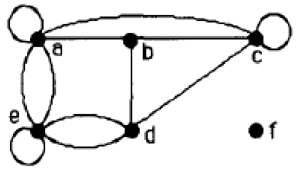
9. This directed multigraph has 5 vertices and 13 edges. The in-degree of vertex a is $\deg^-(a) = 6$ since there are 6 edges with a as their terminal vertex; its out-degree is $\deg^+(a) = 1$. Similarly we have $\deg^-(b) = 1$, $\deg^+(b) = 5$, $\deg^-(c) = 2$, $\deg^+(c) = 5$, $\deg^-(d) = 4$, $\deg^+(d) = 2$, $\deg^-(e) = 0$, and $\deg^+(e) = 0$ (vertex e is isolated). As a check we see that the sum of the in-degrees and the sum of the out-degrees are both equal to the number of edges (13).

10. For each of the graphs in Exercises 7–9 determine the sum of the in-degrees of the vertices and the sum of the out-degrees of the vertices directly. Show that they are both equal to the number of edges in the graph.

10. For Exercise 7 the sum of the in-degrees is $3+1+2+1 = 7$, and the sum of the out-degrees is $1+2+1+3 = 7$; there are 7 edges. For Exercise 8 the sum of the in-degrees is $2+3+2+1 = 8$, and the sum of the out-degrees is $2+4+1+1 = 8$; there are 8 edges. For Exercise 9 the sum of the in-degrees is $6+1+2+4+0 = 13$, and the sum of the out-degrees is $1+5+5+2+0 = 13$; there are 13 edges.

11. Construct the underlying undirected graph for the graph with directed edges in Figure 2.

11. To form the underlying undirected graph we simply take all the arrows off the edges. Thus, for example, the edges from e to d and from d to e become a pair of parallel edges between e and d.



---

12. What does the degree of a vertex represent in the acquaintanceship graph, where vertices represent all the people in the world? What does the neighborhood a vertex in this graph represent? What do isolated and pendant vertices in this graph represent? In one study it was estimated that the average degree of a vertex in this graph is 1000. What does this mean in terms of the model?

12. Since there is an edge from a person to each of his or her acquaintances, the degree of v is the number of people v knows. An isolated vertex would be a person who knows no one, and a pendant vertex would be a person who knows just one other person (it is doubtful that there are many, if any, isolated or pendant vertices). If the average degree is 1000, then the average person knows 1000 other people.

---

13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices

13. Since a person is joined by an edge to each of his or her collaborators, the degree of v is the number of collaborators v has. Similarly, the neighborhood of a vertex is the set of coauthors of the person represented by that vertex. An isolated vertex represents a person who has no coauthors (he or she has published only single-authored papers), and a pendant vertex represents a person who has published with just one other person.

---

14. What does the degree of a vertex in the Hollywood graph represent? What does the neighborhood of a vertex represent? What do the isolated and pendant vertices represent?

14. Since there is an edge from a person to each of the other actors with whom that person has appeared in a movie, the degree of v is the number of other actors with whom that person has appeared. The neighborhood of v is the *set* of actors with whom v as appeared. An isolated vertex would be a person who has appeared only in movies in which he or she was the only actor, and a pendant vertex would be a person who has appeared with only one other actor in any movie (it is doubtful that there are many, if any, isolated or pendant vertices).

---

15. What do the in-degree and the out-degree of a vertex in a telephone call graph, as described in Example 4 of Section 10.1, represent? What does the degree of a vertex in the undirected version of this graph represent?

15. Since there is a directed edge from u to v for each call made by u to v, the in-degree of v is the number of calls v received, and the out-degree of u is the number of calls u made. The degree of a vertex in the undirected version is just the sum of these, which is therefore the number of calls the vertex was involved in.

---

16. What do the in-degree and the out-degree of a vertex in the Web graph, as described in Example 5 of Section 10.1, represent?

16. Since there is an edge from a page to each page that it links to, the outdegree of a vertex is the number of links on that page, and the in-degree of a vertex is the number of other pages that have a link to it.

---

17. What do the in-degree and the out-degree of a vertex in a directed graph modeling a round-robin tournament represent?

17. Since there is a directed edge from u to v to represent the event that u beat v when they played, the in-degree of v must be the number of teams that beat v, and the out-degree of u must be the number of teams that u beat. In other words, the pair $(\deg^+(v), \deg^-(v))$ is the win-loss record of v.

---

18. Show that in a simple graph with at least two vertices there must be two vertices that have the same degree.

18. This is essentially the same as Exercise 40 in Section 6.2, where the graph models the "know each other" relation on the people at the party. See the solution given for that exercise. The number of people a person knows is the degree of the corresponding vertex in the graph.

---

19. Use Exercise 18 to show that in a group of people, there must be two people who are friends with the same number of other people in the group.

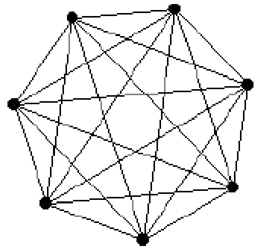
19. Model the friendship relation with a simple undirected graph in which the vertices are people in the group, and two vertices are adjacent if those two people are friends. The degree of a vertex is the number of friends in the group that person has. By Exercise 18, there are two vertices with the same degree, which means that there are two people in the group with the same number of friends in the group.

---

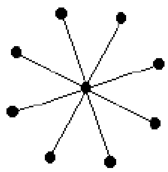
20. Draw these graphs.

a) K_7 **b)** $K_{1,8}$ **c)** $K_{4,4}$
d) C_7 **e)** W_7 **f)** Q_4

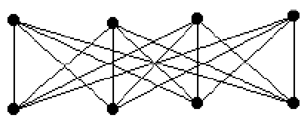
20. a) This graph has 7 vertices, with an edge joining each pair of distinct vertices.



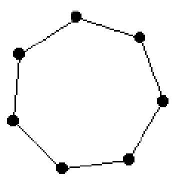
b) This graph is the complete bipartite graph on parts of size 1 and 8; we have put the part of size 1 in the middle.



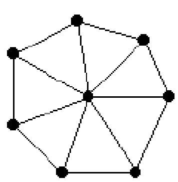
c) This is the complete bipartite graph with 4 vertices in each part.



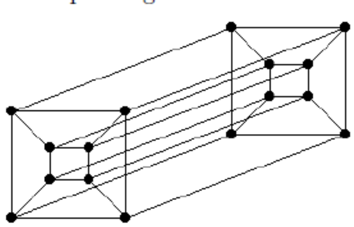
d) This is the 7-cycle.



e) The 7-wheel is the 7-cycle with an extra vertex joined to the other 7 vertices. Warning: Some texts call this W_8, to have the consistent notation that the subscript in the name of a graph should be the number of vertices in that graph.

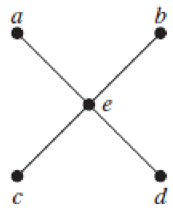


f) We take two copies of Q_3 and join corresponding vertices.

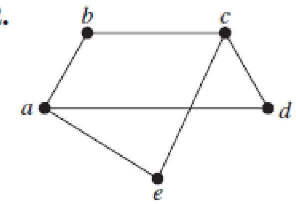


In Exercises 21–25 determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

21.



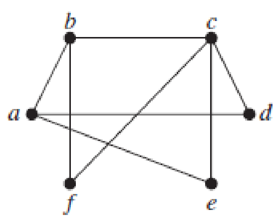
22.



21. To show that this graph is bipartite we can exhibit the parts and note that indeed every edge joins vertices in different parts. Take $\{e\}$ to be one part and $\{a, b, c, d\}$ to be the other (in fact there is no choice in the matter). Each edge joins a vertex in one part to a vertex in the other. This graph is the complete bipartite graph $K_{1,4}$.

22. This graph is bipartite, with bipartition $\{a, c\}$ and $\{b, d, e\}$. In fact this is the complete bipartite graph $K_{2,3}$. If this graph were missing the edge between a and d, then it would still be bipartite on the same sets, but not a complete bipartite graph.

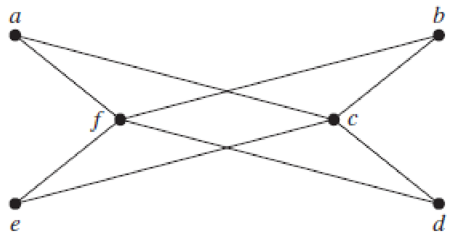
23.



23. To show that a graph is not bipartite we must give a proof that there is no possible way to specify the parts. (There is another good way to characterize nonbipartite graphs, but it takes some notions not introduced until Section 10.4.) We can show that this graph is not bipartite by the pigeonhole principle. Consider the vertices b, c, and f. They form a triangle—each is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

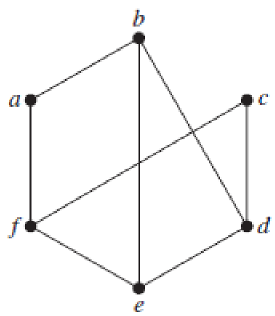
An alternative way to look at this is given by Theorem 4. Because of the triangle, it is impossible to color the vertices to satisfy the condition given there.

24.



24. This is the complete bipartite graph $K_{2,4}$. The vertices in the part of size 2 are c and f, and the vertices in the part of size 4 are a, b, d, and e.

25.



25. As in Exercise 23, we can show that this graph is not bipartite by looking at a triangle, in this case the triangle formed by vertices b, d, and e. Each of these vertices is joined by an edge to the other two. By the pigeonhole principle, at least two of them must be in the same part of any proposed bipartition. Therefore there would be an edge joining two vertices in the same part, a contradiction to the definition of a bipartite graph. Thus this graph is not bipartite.

---

26. For which values of n are these graphs bipartite?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

26. a) By the definition given in the text, K_1 does not have enough vertices to be bipartite (the sets in a partition have to be nonempty). Clearly K_2 is bipartite. There is a triangle in K_n for $n > 2$, so those complete graphs are not bipartite. (See Exercise 23.)

b) First we need $n \geq 3$ for C_n to be defined. If n is even, then C_n is bipartite, since we can take one part to be every other vertex. If n is odd, then C_n is not bipartite.

c) Every wheel contains triangles, so no W_n is bipartite.

d) Q_n is bipartite for all $n \geq 1$, since we can divide the vertices into these two classes: those bit strings with an odd number of 1's, and those bit strings with an even number of 1's.

---

27. Suppose that there are four employees in the computer support group of the School of Engineering of a large university. Each employee will be assigned to support one of four different areas: hardware, software, networking, and wireless. Suppose that Ping is qualified to support hardware, networking, and wireless; Quiggley is qualified to support software and networking; Ruiz is qualified to support networking and wireless, and Sitea is qualified to support hardware and software.

a) Use a bipartite graph to model the four employees and their qualifications.

b) Use Hall's theorem to determine whether there is an assignment of employees to support areas so that each employee is assigned one area to support.

c) If an assignment of employees to support areas so that each employee is assigned to one support area exists, find one.

27. a) The bipartite graph has vertices h, s, n, and w representing the support areas and P, Q, R, and S representing the employees. The qualifications are modeled by the bipartite graph with edges Ph, Pn, Pw, Qs, Qn, Rn, Rw, Sh, and Ss.

b) Since every vertex representing an area has degree at least 2, the condition in Hall's theorem is satisfied for sets of size less than 3. We can easily check that the number of employees qualified for each of the four subsets of size 3 is at least 3, and clearly the number of employees qualified for each of the subsets of size 4 has size 4.

c) The answer is not unique; one complete matching is $\{Pn, Qs, Rw, Sh\}$, which is easily found by inspection.

28. Suppose that a new company has five employees: Zamora, Agraharam, Smith, Chou, and Macintyre. Each employee will assume one of six responsiblities: planning, publicity, sales, marketing, development, and industry relations. Each employee is capable of doing one or more of these jobs: Zamora could do planning, sales, marketing, or industry relations; Agraharam could do planning or development; Smith could do publicity, sales, or industry relations; Chou could do planning, sales, or industry relations; and Macintyre could do planning, publicity, sales, or industry relations.

a) Model the capabilities of these employees using a bipartite graph.

b) Find an assignment of responsibilites such that each employee is assigned one responsibility.

c) Is the matching of responsibilities you found in part (b) a complete matching? Is it a maximum matching?

28. a) Following the lead in Example 14, we construct a bipartite graph in which the vertex set consists of two subsets—one for the employees and one for the jobs. Let $V_1 = \{\text{Zamora}, \text{Agraharam}, \text{Smith}, \text{Chou}, \text{Macintyre}\}$, and let $V_2 = \{\text{planning}, \text{publicity}, \text{sales}, \text{marketing}, \text{development}, \text{industry relations}\}$. Then the vertex set for our graph is $V = V_1 \cup V_2$. Given the list of capabilities in the exercise, we must include precisely the following edges in our graph: {Zamora, planning}, {Zamora, sales}, {Zamora, marketing}, {Zamora, industry relations}, {Agraharam, planning}, {Agraharam, development}, {Smith, publicity}, {Smith, sales}, {Smith, industry relations}, {Chou, planning}, {Chou, sales}, {Chou, industry relations}, {Macintyre, planning}, {Macintyre, publicity}, {Macintyre, sales}, {Macintyre, industry relations}.

b) Many assignments are possible. If we take it as an implicit assumption that there will be no more than one employee assigned to the same job, then we want a maximum matching for this graph. So we look for five edges in this graph that share no endpoints. A little trial and error gives us, for example, {Zamora, planning}, {Agraharam, development}, {Smith, publicity}, {Chou, sales}, {Macintyre, industry relations}. We assign the employees to the jobs given in this matching.

c) This is a complete matching from the set of employees to the set of jobs, but not the other way around. It is a maximum matching; because there were only five employees, no matching could have more than five edges.

29. Suppose that there are five young women and five young men on an island. Each man is willing to marry some of the women on the island and each woman is willing to marry any man who is willing to marry her. Suppose that Sandeep is willing to marry Tina and Vandana; Barry is willing to marry Tina, Xia, and Uma; Teja is willing to marry Tina and Zelda; Anil is willing to marry Vandana and Zelda; and Emilio is willing to marry Tina and Zelda. Use Hall's theorem to show there is no matching of the young men and young women on the island such that each young man is matched with a young woman he is willing to marry.

29. The partite sets are the set of women ($\{\text{Tina, Uma, Vandana, Xia, Zelda}\}$) and the set of men ($\{\text{Anil, Barry, Emilio, Sandeep, Teja}\}$). We will use first letters for convenience (but J for Teja). The given information tells us that we have edges AV, AZ, BT, BX, BU, ET, EZ, JT, JZ, ST, and SV in our graph. We do not put an edge between a man and a woman he is not willing to marry. By inspection we find that the condition in Hall's theorem is violated by $\{U, X\}$, because these two vertices are adjacent only to B. In other words, only Barry is willing to marry Uma and Xia, so there can be no matching.

---

30. Suppose that there are five young women and six young men on an island. Each woman is willing to marry some of the men on the island and each man is willing to marry any woman who is willing to marry him. Suppose that Anna is willing to marry Jason, Larry, and Matt; Barbara is willing to marry Kevin and Larry; Carol is willing to marry Jason, Nick, and Oscar; Diane is willing to marry Jason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt.

a) Model the possible marriages on the island using a bipartite graph.

b) Find a matching of the young women and the young men on the island such that each young woman is matched with a young man whom she is willing to marry.

c) Is the matching you found in part (b) a complete matching? Is it a maximum matching?

30. a) The partite sets are the set of women ($\{\text{Anna, Barbara, Carol, Diane, Elizabeth}\}$) and the set of men ($\{\text{Jason, Kevin, Larry, Matt, Nick, Oscar}\}$). We will use first letters for convenience. The given information tells us to have edges AJ, AL, AM, BK, BL, CJ, CN, CO, DJ, DL, DN, DO, EJ, and EM in our graph. We do not put an edge between a woman and a man she is not willing to marry.

b) By trial and error we easily find a matching (it's not unique), such as AL, BK, CJ, DN, and EM.

c) This is a complete matching from the women to the men (as well as from the men to the women). A complete matching is always a maximum matching.

***31.** Suppose there is an integer k such that every man on a desert island is willing to marry exactly k of the women on the island and every woman on the island is willing to marry exactly k of the men. Also, suppose that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match the men and women on the island so that everyone is matched with someone that they are willing to marry.

31. We model this with an undirected bipartite graph, with the men and the women represented by the vertices in the two parts and an edge between two vertices if they are willing to marry each other. By Hall's theorem, it is enough to show that for every set S of women, the set $N(S)$ of men willing to marry them has cardinality at least $|S|$. A clever way to prove this is by counting edges. Let m be the number of edges between S and $N(S)$. Since every vertex in S has degree k, it follows that $m = k|S|$. Because these edges are incident to $N(S)$, it follows that $m \leq k|N(S)|$. Combining these two facts gives $k|S| \leq k|N(S)|$, so $|N(S)| \geq |S|$, as desired.

***32.** In this exercise we prove a theorem of Øystein Ore. Suppose that $G = (V, E)$ is a bipartite graph with bipartition (V_1, V_2) and that $A \subseteq V_1$. Show that the maximum number of vertices of V_1 that are the endpoints of a matching of G equals $|V_1| - \max_{A \subseteq V_1} \text{def}(A)$, where $\text{def}(A) = |A| - |N(A)|$. (Here, $\text{def}(A)$ is called the **deficiency** of A.) [*Hint:* Form a larger graph by adding $\max_{A \subseteq V_1} \text{def}(A)$ new vertices to V_2 and connect all of them to the vertices of V_1.]

32. Let $d = \max_{A \subseteq V_1} \text{def}(A)$, and fix A to be a subset of V_1 that achieves this maximum. Thus $d = |A| - |N(A)|$. First we show that no matching in G can touch more than $|V_1| - d$ vertices of V_1 (or, equivalently, that no matching in G can have more than $|V_1| - d$ edges). At most $|N(A)|$ edges of such a matching can have endpoints in A, and at most $|V_1| - |A|$ can have endpoints in $V_1 - A$, so the total number of such edges is at most $|N(A)| + |V_1| - |A| = |V_1| - d$. It remains to show that we can find a matching in G touching (at least) $|V_1| - d$ vertices of V_1 (i.e., a matching in G with $|V_1| - d$ edges). Following the hint, construct a larger graph G' by adding d new vertices to V_2 and joining all of them to all the vertices of V_1. Then the condition in Hall's theorem holds in G', so G' has a matching that touches all the vertices of V_1. At most d of these edges do not lie in G, and so the edges of this matching that do lie in G form a matching in G with at least $|V_1| - d$ edges.

33. For the graph G in Exercise 1 find

a) the subgraph induced by the vertices a, b, c, and f.

b) the new graph G_1 obtained from G by contracting the edge connecting b and f.

33. a) By definition, the vertices are a, b, c, and f, and the edges are all the edges of the given graph joining vertices in this list, namely ab, af, bc, and bf.

b) Contracting edge bf merges the vertices b and f into a new vertex; call it x. Edges ab and af are replaced by edge ax; edges eb and ef are replaced by edge ex; and edge cb is replaced by edge cx. Vertex d continues to be an isolated vertex in the contracted graph.

---

34. Let n be a positive integer. Show that a subgraph induced by a nonempty subset of the vertex set of K_n is a complete graph.

34. Since all the vertices in the subgraph are adjacent in K_n, they are adjacent in the subgraph, i.e., the subgraph is complete.

---

35. How many vertices and how many edges do these graphs have?

a) K_n **b)** C_n **c)** W_n

d) $K_{m,n}$ **e)** Q_n

The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order. For example, the degree sequence of the graph G in Example 1 is 4, 4, 4, 3, 2, 1, 0.

35. a) Obviously K_n has n vertices. It has $C(n,2) = n(n-1)/2$ edges, since each unordered pair of distinct vertices is an edge.

b) Obviously C_n has n vertices. Just as obviously it has n edges.

c) The wheel W_n is the same as C_n with an extra vertex and n extra edges incident to that vertex. Therefore it has $n+1$ vertices and $n+n=2n$ edges.

d) By definition $K_{m,n}$ has $m+n$ vertices. Since it has one edge for each choice of a vertex in the one part and a vertex in the other part, it has mn edges.

e) Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the handshaking theorem), we know that there are $n2^n/2 = n2^{n-1}$ edges.

36. Find the degree sequences for each of the graphs in Exercises 21–25.

36. We just have to count the number of edges at each vertex, and then arrange these counts in nonincreasing order. For Exercise 21, we have $4, 1, 1, 1, 1$. For Exercise 22, we have $3, 3, 2, 2, 2$. For Exercise 23, we have $4, 3, 3, 2, 2, 2$. For Exercise 24, we have $4, 4, 2, 2, 2, 2$. For Exercise 25, we have $3, 3, 3, 3, 2, 2$.

---

37. Find the degree sequence of each of the following graphs.

a) K_4 **b)** C_4 **c)** W_4
d) $K_{2,3}$ **e)** Q_3

37. In each case we just record the degrees of the vertices in a list, from largest to smallest.

a) Each of the four vertices is adjacent to each of the other three vertices, so the degree sequence is $3, 3, 3, 3$.

b) Each of the four vertices is adjacent to its two neighbors in the cycle, so the degree sequence is $2, 2, 2, 2$.

c) Each of the four vertices on the rim of the wheel is adjacent to each of its two neighbors on the rim, as well as to the middle vertex. The middle vertex is adjacent to the four rim vertices. Therefore the degree sequence is $4, 3, 3, 3, 3$.

d) Each of the vertices in the part of size two is adjacent to each of the three vertices in the part of size three, and vice versa, so the degree sequence is $3, 3, 2, 2, 2$.

e) Each of the eight vertices in the cube is adjacent to three others (for example, 000 is adjacent to 001, 010, and 100. Therefore the degree sequence is $3, 3, 3, 3, 3, 3, 3, 3$.

---

38. What is the degree sequence of the bipartite graph $K_{m,n}$ where m and n are positive integers? Explain your answer.

38. Assume that $m \geq n$. Then each of the n vertices in one part has degree m, and each of the m vertices in other part has degree n. Thus the degree sequence is $m, m, \ldots, m, n, n, \ldots, n$, where the sequence contains n copies of m and m copies of n. We put the m's first because we assumed that $m \geq n$. If $n \geq m$, then of course we would put the m copies of n first. If $m = n$, this would mean a total of $2n$ copies of n.

---

39. What is the degree sequence of K_n, where n is a positive integer? Explain your answer.

39. Each of the n vertices is adjacent to each of the other $n-1$ vertices, so the degree sequence is simply $n-1, n-1, \ldots, n-1$, with n terms in the sequence.

---

40. How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

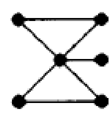
40. The 4-wheel (see Figure 5) with one edge along the rim deleted is such a graph. It has $(4+3+3+2+2)/2 = 7$ edges.

---

41. How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

41. The number of edges is half the sum of the degrees (Theorem 1). Therefore this graph has $(5+2+2+2+2+1)/2 = 7$ edges. A picture of this graph is shown here (it is essentially unique).



---

42. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 5, 4, 3, 2, 1, 0 **b)** 6, 5, 4, 3, 2, 1 **c)** 2, 2, 2, 2, 2, 2
d) 3, 3, 3, 2, 2, 2 **e)** 3, 3, 2, 2, 2, 2 **f)** 1, 1, 1, 1, 1, 1
g) 5, 3, 3, 3, 3, 3 **h)** 5, 5, 4, 3, 2, 1

42. a) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the vertex of degree 0 would have to be isolated but the vertex of degree 5 would have to be adjacent to every other vertex, and these two statements are contradictory.

b) Since the number of odd-degree vertices has to be even, no graph exists with these degrees. Another reason no such graph exists is that the degree of a vertex in a simple graph is at most 1 less than the number of vertices.

c) A 6-cycle is such a graph. (See picture below.)

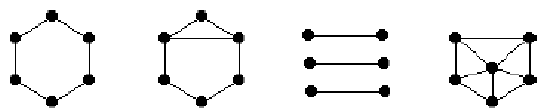
d) Since the number of odd-degree vertices has to be even, no graph exists with these degrees.

e) A 6-cycle with one of its diagonals added is such a graph. (See picture below.)

f) A graph consisting of three edges with no common vertices is such a graph. (See picture below.)

g) The 5-wheel is such a graph. (See picture below.)

h) Each of the vertices of degree 5 is adjacent to all the other vertices. Thus there can be no vertex of degree 1. So no such graph exists.



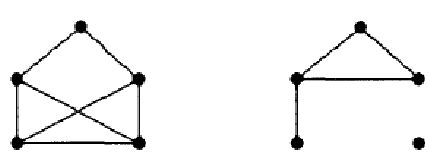
(c) (e) (f) (g)

---

43. Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

a) 3, 3, 3, 3, 2 **b)** 5, 4, 3, 2, 1 **c)** 4, 4, 3, 2, 1
d) 4, 4, 3, 3, 3 **e)** 3, 2, 2, 1, 0 **f)** 1, 1, 1, 1, 1

43. There is no such graph in part (b), since the sum of the degrees is odd (and also because a simple graph with 5 vertices cannot have any degrees greater than 4). Similarly, the odd degree sum prohibits the existence of graphs with the degree sequences given in part (d) and part (f). There is no such graph in part (c), since the existence of two vertices of degree 4 implies that there are two vertices each joined by an edge to every other vertex. This means that the degree of each vertex has to be at least 2, and there can be no vertex of degree 1. The graphs for part (a) and part (e) are shown below; one can draw them after just a little trial and error.



(a) (e)

***44.** Suppose that $d_1, d_2, \ldots, d_n$ is a graphic sequence. Show that there is a simple graph with vertices $v_1, v_2, \ldots, v_n$ such that $\deg(v_i) = d_i$ for $i = 1, 2, \ldots, n$ and v_1 is adjacent to $v_2, \ldots, v_{d_1+1}$.

44. Since isolated vertices play no essential role, we can assume that $d_n > 0$. The sequence is graphic, so there is some simple graph G such that the degrees of the vertices are $d_1, d_2, \ldots, d_n$. Without loss of generality, we can label the vertices of our graph so that $d(v_i) = d_i$. Among all such graphs, choose G to be one in which v_1 is adjacent to as many of $v_2, v_3, \ldots, v_{d_1+1}$ as possible. (The worst case might be that v_1 is not adjacent to any of these vertices.) If v_1 is adjacent to all of them, then we are done. We will show that if there is a vertex among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is not adjacent to, then we can find another graph with $d(v_i) = d_i$ and having v_1 adjacent to one more of the vertices $v_2, v_3, \ldots, v_{d_1+1}$ than is true for G. This is a contradiction to the choice of G, and hence we will have shown that G satisfies the desired condition.

Under this assumption, then, let u be a vertex among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is not adjacent to, and let w be a vertex not among $v_2, v_3, \ldots, v_{d_1+1}$ that v_1 is adjacent to; such a vertex w has to exist because $d(v_1) = d_1$. Because the degree sequence is listed in nonincreasing order, we have $d(u) \geq d(w)$. Consider all the vertices that are adjacent to u. It cannot be the case that w is adjacent to each of them, because then w would have a higher degree than u (because w is adjacent to v_1 as well, but u is not). Therefore there is some vertex x such that edge ux is present but edge xw is not present. Note also that edge v_1w is present but edge v_1u is not present. Now construct the graph G' to be the same as G except that edges ux and v_1w are removed and edges xw and v_1u are added. The degrees of all vertices are unchanged, but this graph has v_1 adjacent to more of the vertices among $v_2, v_3, \ldots, v_{d_1+1}$ than is the case in G. That gives the desired contradiction, and our proof is complete.

---

48. How many subgraphs with at least one vertex does K_2 have?

48. We list the subgraphs: the subgraph consisting of K_2 itself, the subgraph consisting of two vertices and no edges, and two subgraphs with 1 vertex each. Therefore the answer is 4.

---

49. How many subgraphs with at least one vertex does K_3 have?

49. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 3 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(3,2) = 3$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $3 \cdot 2 = 6$ subgraphs with two vertices. If a subgraph is to have all three vertices, then there are $2^3 = 8$ ways to decide whether or not to include each of the edges. Thus our answer is $3 + 6 + 8 = 17$.

50. How many subgraphs with at least one vertex does W_3 have?

50. We need to count this in an organized manner. First note that W_3 is the same as K_4, and it will be easier if we think of it as K_4. We will count the subgraphs in terms of the number of vertices they contain. There are clearly just 4 subgraphs consisting of just one vertex. If a subgraph is to have two vertices, then there are $C(4,2) = 6$ ways to choose the vertices, and then 2 ways in each case to decide whether or not to include the edge joining them. This gives us $6 \cdot 2 = 12$ subgraphs with two vertices. If a subgraph is to have three vertices, then there are $C(4,3) = 4$ ways to choose the vertices, and then $2^3 = 8$ ways in each case to decide whether or not to include each of the edges joining pairs of them. This gives us $4 \cdot 8 = 32$ subgraphs with three vertices. Finally, there are the subgraphs containing all four vertices. Here there are $2^6 = 64$ ways to decide which edges to include. Thus our answer is $4 + 12 + 32 + 64 = 112$.

52. Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices of G, and let m be the minimum degree of the vertices of G. Show that

a) $2e/v \geq m$. **b)** $2e/v \leq M$.

52. a) We want to show that $2e \geq vm$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot be less than the sum of m for each vertex, since each degree is no less than m.
b) We want to show that $2e \leq vM$. We know from Theorem 1 that $2e$ is the sum of the degrees of the vertices. This certainly cannot exceed the sum of M for each vertex, since each degree is no greater than M.

A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called n-**regular** if every vertex in this graph has degree n.

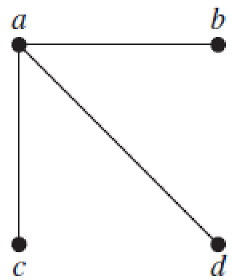
53. For which values of n are these graphs regular?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

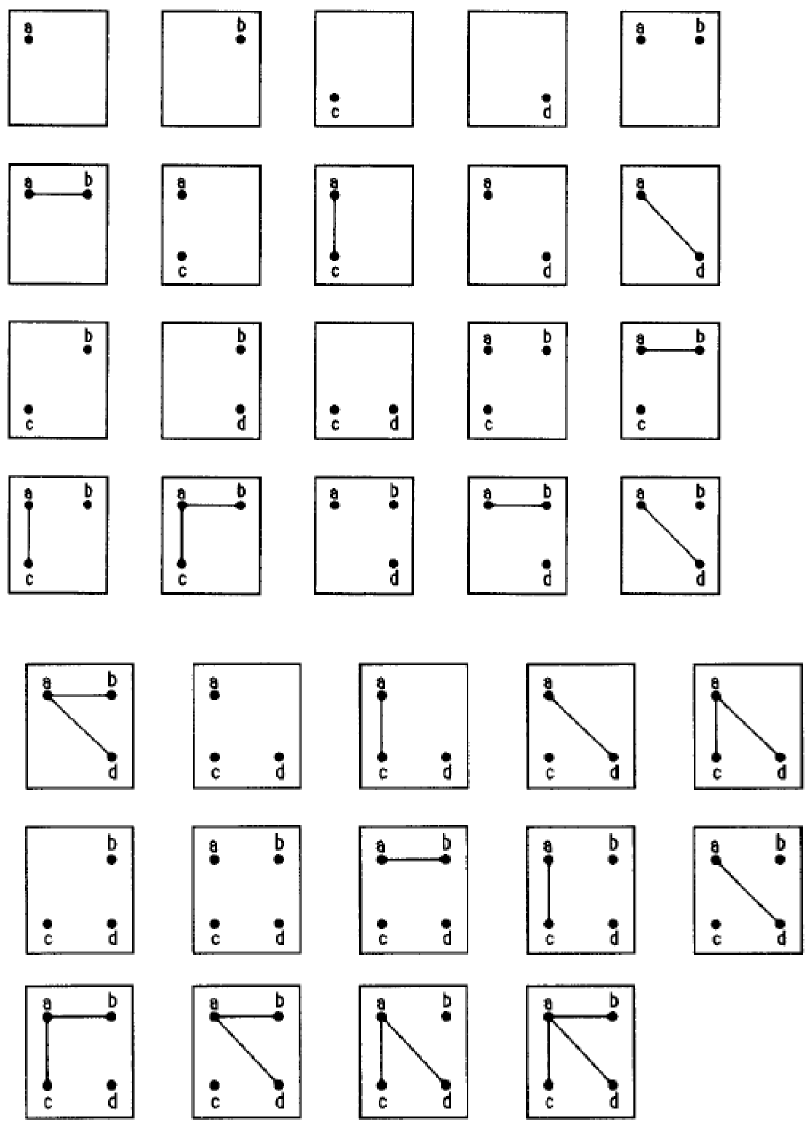
53. a) The complete graph K_n is regular for all values of $n \geq 1$, since the degree of each vertex is $n-1$.
b) The degree of each vertex of C_n is 2 for all n for which C_n is defined, namely $n \geq 3$, so C_n is regular for all these values of n.
c) The degree of the middle vertex of the wheel W_n is n, and the degree of the vertices on the "rim" is 3. Therefore W_n is regular if and only if $n = 3$. Of course W_3 is the same as K_4.
d) The cube Q_n is regular for all values of $n \geq 0$, since the degree of each vertex in Q_n is n. (Note that Q_0 is the graph with 1 vertex.)

---

51. Draw all subgraphs of this graph.



51. This graph has a lot of subgraphs. First of all, any nonempty subset of the vertex set can be the vertex set for a subgraph, and there are 15 such subsets. If the set of vertices of the subgraph does not contain vertex a, then the subgraph can of course have no edges. If it does contain vertex a, then it can contain or fail to contain each edge from a to whichever other vertices are included. A careful enumeration of all the possibilities gives the 34 graphs shown below.



54. For which values of m and n is $K_{m,n}$ regular?

54. Since the vertices in one part have degree m, and vertices in the other part have degree n, we conclude that $K_{m,n}$ is regular if and only if $m = n$.

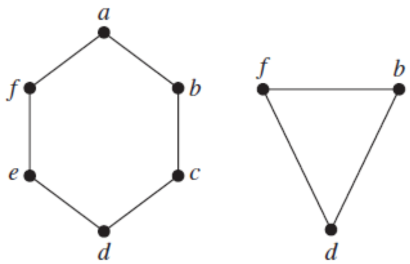
---

55. How many vertices does a regular graph of degree four with 10 edges have?

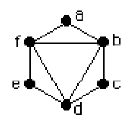
55. If a graph is regular of degree 4 and has n vertices, then by the handshaking theorem it has $4n/2 = 2n$ edges. Since we are told that there are 10 edges, we just need to solve $2n = 10$. Thus the graph has 5 vertices. The complete graph K_5 is one such graph (and the only simple one).

In Exercises 56–58 find the union of the given pair of simple graphs. (Assume edges with the same endpoints are the same.)

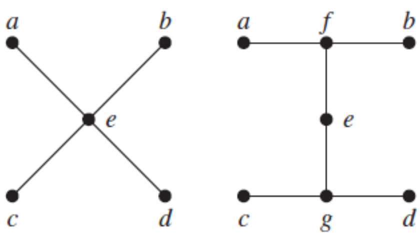
56.



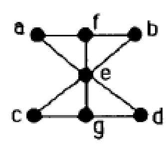
56. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



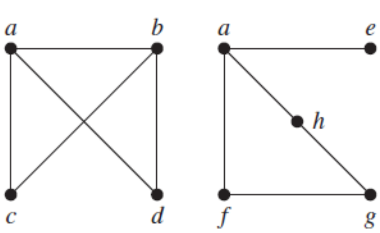
57.



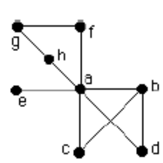
57. We draw the answer by superimposing the graphs (keeping the positions of the vertices the same).



58.



58. The union is shown here. The only common vertex is a, so we have reoriented the drawing so that the pieces will not overlap.



59. The **complementary graph** $\overline{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Describe each of these graphs.

a) $\overline{K_n}$ **b)** $\overline{K_{m,n}}$ **c)** $\overline{C_n}$ **d)** $\overline{Q_n}$

59. a) The complement of a complete graph is a graph with no edges.

b) Since all the edges between the parts are present in $K_{m,n}$, but none of the edges between vertices in the same part are, the complement must consist precisely of the disjoint union of a K_m and a K_n, i.e., the graph containing all the edges joining two vertices in the same part and no edges joining vertices in different parts.

c) There is really no better way to describe this graph than simply by saying it is the complement of C_n. One representation would be to take as vertex set the integers from 1 to n, inclusive, with an edge between distinct vertices i and j as long as i and j do not differ by ± 1, modulo n.

d) Again, there is really no better way to describe this graph than simply by saying it is the complement of Q_n. One representation would be to take as vertex set the bit strings of length n, with two vertices joined by an edge if the bit strings differ in more than one bit.

---

60. If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

60. The given information tells us that $G \cup \overline{G}$ has 28 edges. However, $G \cup \overline{G}$ is the complete graph on the number of vertices n that G has. Since this graph has $n(n-1)/2$ edges, we want to solve $n(n-1)/2 = 28$. Thus $n = 8$.

---

61. If the simple graph G has v vertices and e edges, how many edges does $\overline{G}$ have?

61. Since K_v has $C(v,2) = v(v-1)/2$ edges, and since $\overline{G}$ has all the edges of K_v that G is missing, it is clear that $\overline{G}$ has $[v(v-1)/2] - e$ edges.

---

62. If the degree sequence of the simple graph G is 4, 3, 3, 2, 2, what is the degree sequence of $\overline{G}$?

62. Following the ideas given in the solution to Exercise 63, we see that the degree sequence is obtained by subtracting each of these numbers from 4 (the number of vertices) and reversing the order. We obtain $2, 2, 1, 1, 0$.

---

63. If the degree sequence of the simple graph G is $d_1, d_2, \ldots, d_n$, what is the degree sequence of $\overline{G}$?

63. If G has n vertices, then the degree of vertex v in $\overline{G}$ is $n-1$ minus the degree of v in G (there will be an edge in $\overline{G}$ from v to each of the $n-1$ other vertices that v is not adjacent to in G). The order of the sequence will reverse, of course, because if $d_i \geq d_j$, then $n-1-d_i \leq n-1-d_j$. Therefore the degree sequence of $\overline{G}$ will be $n-1-d_n$, $n-1-d_{n-1}$, ..., $n-1-d_2$, $n-1-d_1$.

***64.** Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq v^2/4$.

64. Suppose the parts are of sizes k and $v-k$. Then the maximum number of edges the graph may have is $k(v-k)$ (an edge between each pair of vertices in different parts). By algebra or calculus, we know that the function $f(k) = k(v-k)$ achieves its maximum when $k = v/2$, giving $f(k) = v^2/4$. Thus there are at most $v^2/4$ edges.

The **converse** of a directed graph $G = (V, E)$, denoted by G^{conv}, is the directed graph (V, F), where the set F of edges of G^{conv} is obtained by reversing the direction of each edge in E.

67. Draw the converse of each of the graphs in Exercises 7–9 in Section 10.1.

67. These pictures are identical to the figures in those exercises, with one change, namely that all the arrowheads are turned around. For example, rather than there being a directed edge from a to b in #7, there is an edge from b to a. Note that the loops are unaffected by changing the direction of the arrowhead—a loop from a vertex to itself is the same, whether the drawing of it shows the direction to be clockwise or counterclockwise.

SECTION 10.3 Representing Graphs and Graph Isomorphism

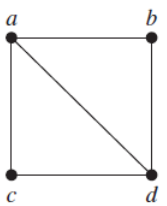
Human beings can get a good feeling for a small graph by looking at a picture of it drawn with points in the plane and lines or curves joining pairs of these points. If a graph is at all large (say with more than a dozen vertices or so), then the picture soon becomes too crowded to be useful. A computer has little use for nice pictures, no matter how small the vertex set. Thus people and machines need more precise—more discrete—representations of graphs. In this section we learned about some useful representations. They are for the most part exactly what any intelligent person would come up with, given the assignment to do so.

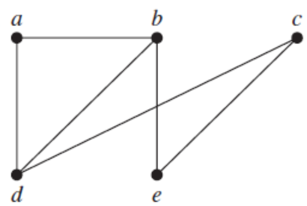
The only tricky idea in this section is the concept of graph isomorphism. It is a special case of a more general notion of isomorphism, or sameness, of mathematical objects in various settings. Isomorphism tries to capture the idea that all that really matters in a graph is the adjacency structure. If we can find a way to superimpose the graphs so that the adjacency structures match, then the graphs are, for all purposes that matter, the same. In trying to show that two graphs are isomorphic, try moving the vertices around in your mind to see whether you can make the graphs look the same. Of course there are often lots of things to help. For example, in every isomorphism, vertices that correspond must have the same degree.

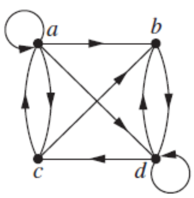
A good general strategy for determining whether two graphs are isomorphic might go something like this. First check the degrees of the vertices to make sure there are the same number of each degree. See whether vertices of corresponding degrees follow the same adjacency pattern (e.g., if there is a vertex of degree 1 adjacent to a vertex of degree 4 in one of the graphs, then there must be the same pattern in the other, if the graphs are isomorphic). Then look for triangles in the graphs, and see whether they correspond. Sometimes, if the graphs have lots of edges, it is easier to see whether the complements are isomorphic (see Exercise 46). If you cannot find a good reason for the graphs not to be isomorphic (an invariant on which they differ), then try to write down a one-to-one and onto function that shows them to be isomorphic (there may be more than one such function); such a function has to have vertices of like degrees correspond, so often the function practically writes itself. Then check each edge of the first graph to make sure that it corresponds to an edge of the second graph under this correspondence.

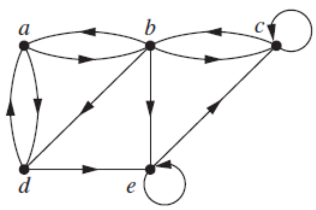
Unfortunately, no one has yet discovered a really good algorithm for determining graph isomorphism that works on all pairs of graphs. Research in this subject has been quite active in recent years. See Writing Project 10.

In Exercises 1–4 use an adjacency list to represent the given graph.

1. 

2. 

3. 

4. 

1. Adjacency lists are lists of lists. The adjacency list of an undirected graph is simply a list of the vertices of the given graph, together with a list of the vertices adjacent to each. The list for this graph is as follows. Since, for instance, b is adjacent to a and d, we list a and d in the row for b.

Vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

2. This is similar to Exercise 1. The list is as follows.

Vertex	Adjacent vertices
a	b, d
b	a, d, e
c	d, e
d	a, b, c
e	b, c

3. To form the adjacency list of a directed graph, we list, for each vertex in the graph, the terminal vertex of each edge that has the given vertex as its initial vertex. The list for this directed graph is as follows. For example, since there are edges from d to each of b, c, and d, we put those vertices in the row for d.

Initial vertex	Terminal vertices
a	a, b, c, d
b	d
c	a, b
d	b, c, d

4. This is similar to Exercise 3. The list is as follows.

Initial vertex	Terminal vertices
a	b, d
b	a, c, d, e
c	b, c
d	a, e
e	c, e

5. Represent the graph in Exercise 1 with an adjacency matrix.

5. For Exercises 5–8 we assume that the vertices are listed in alphabetical order. The matrix contains a 1 as entry (i, j) if there is an edge from vertex i to vertex j; otherwise that entry is 0.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

---

6. Represent the graph in Exercise 2 with an adjacency matrix.

6. This is similar to Exercise 5. The vertices are assumed to be listed in alphabetical order.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

---

7. Represent the graph in Exercise 3 with an adjacency matrix.

7. This is similar to Exercise 5. Note that edges have direction here, so that, for example, the $(1, 2)$ entry is a 1 since there is an edge from a to b, but the $(2, 1)$ entry is a 0 since there is no edge from b to a. Also, the $(1, 1)$ entry is a 1 since there is a loop at a, but the $(2, 2)$ entry is a 0 since there is no loop at b.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

8. Represent the graph in Exercise 4 with an adjacency matrix.

8. This is similar to Exercise 7.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

---

9. Represent each of these graphs with an adjacency matrix.

a) K_4 **b)** $K_{1,4}$ **c)** $K_{2,3}$

d) C_4 **e)** W_4 **f)** Q_3

9. We can solve these problems by first drawing the graph, then labeling the vertices, and finally constructing the matrix by putting a 1 in position (i,j) whenever vertices i and j are joined by an edge. It helps to choose a nice order, since then the matrix will have nice patterns in it.

a) The order of the vertices does not matter, since they all play the same role. The matrix has 0's on the diagonal, since there are no loops in the complete graph.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

b) We put the vertex in the part by itself first.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c) We put the vertices in the part of size 2 first. Notice the block structure.

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

d) We put the vertices in the same order in the matrix as they are around the cycle.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

e) We put the center vertex first. Note that the last four columns of the last four rows represent a C_4.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

f) We can label the vertices by the binary numbers from 0 to 7. Thus the first row (also the first column) of this matrix corresponds to the string 000, the second to the string 001, and so on. Since Q_3 has 8 vertices, this is an 8×8 matrix.

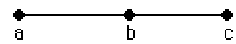
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

In Exercises 10–12 draw a graph with the given adjacency matrix.

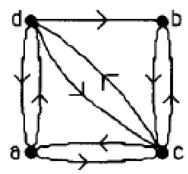
10. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

10. This graph has three vertices and is undirected, since the matrix is symmetric.

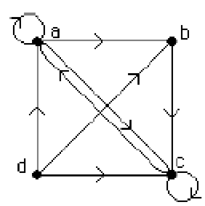


11. This graph has four vertices and is directed, since the matrix is not symmetric. We draw the four vertices as points in the plane, then draw a directed edge from vertex i to vertex j whenever there is a 1 in position (i, j) in the given matrix.

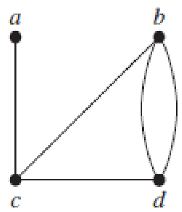


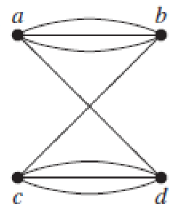
12. $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

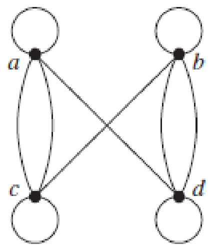
12. This graph is directed, since the matrix is not symmetric.



In Exercises 13–15 represent the given graph using an adjacency matrix.

13. 

14. 

15. 

13. We use alphabetical order of the vertices for Exercises 13–15. If there are k parallel edges between vertices i and j, then we put the number k into the $(i,j)^{\text{th}}$ entry of the matrix. In this exercise, there is only one pair of parallel edges.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

14. This is similar to Exercise 13.

$$\begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

15. This is similar to Exercise 13. In this graph there are loops, which are represented by entries on the diagonal. For example, the loop at c is shown by the 1 as the $(3,3)^{\text{th}}$ entry.

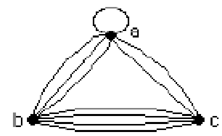
$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

16. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

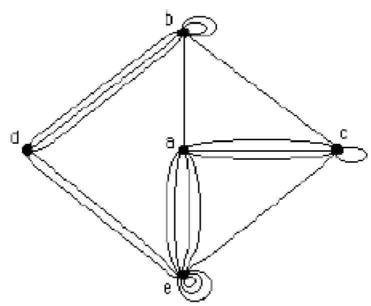
16. Because of the numbers larger than 1, we need multiple edges in this graph.



---

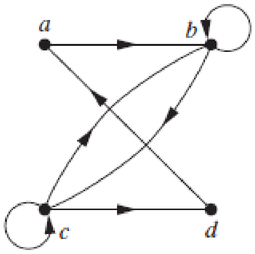
18. $\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$

18. This is similar to Exercise 16.

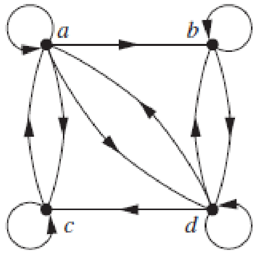


In Exercises 19–21 find the adjacency matrix of the given directed multigraph with respect to the vertices listed in alphabetic order.

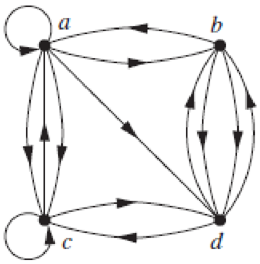
19.



20.



21.



19. We use alphabetical order of the vertices. We put a 1 in position (i, j) if there is a directed edge from vertex i to vertex j; otherwise we make that entry a 0. Note that loops are represented by 1's on the diagonal.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

20. This is similar to Exercise 19.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

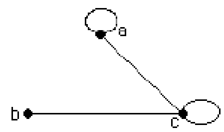
21. This is similar to Exercise 19, except that there are parallel directed edges. If there are k parallel edges from vertex i to vertex j, then we put the number k into the $(i, j)^{\text{th}}$ entry of the matrix. For example, since there are 2 edges from a to c, the $(1, 3)^{\text{th}}$ entry of the adjacency matrix is 2; the loop at c is shown by the 1 as the $(3, 3)^{\text{th}}$ entry.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

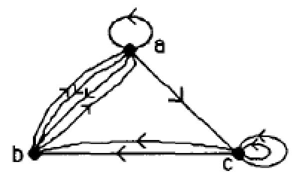
In Exercises 22–24 draw the graph represented by the given adjacency matrix.

22. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **23.** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$ **24.** $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

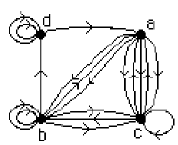
22. a) This matrix is symmetric, so we can take the graph to be undirected. No parallel edges are present, since no entries exceed 1.



23. Since the matrix is not symmetric, we need directed edges; furthermore, it must be a directed multigraph because of the entries larger than 1. For example, the 2 in position $(3, 2)$ means that there are two parallel edges from vertex c to vertex b.



24. This is the adjacency matrix of a directed multigraph, because the matrix is not symmetric and it contains entries greater than 1.



---

25. Is every zero–one square matrix that is symmetric and has zeros on the diagonal the adjacency matrix of a simple graph?

25. Since the matrix is symmetric, it has to be square, so it represents a graph of some sort. In fact, such a matrix does represent a simple graph. The fact that it is a zero-one matrix means that there are no parallel edges. The fact that there are 0's on the diagonal means that there are no loops. The fact that the matrix is symmetric means that the edges can be assumed to be undirected. Note that such a matrix also represents a directed graph in which all the edges happen to appear in antiparallel pairs (see the solution to Exercise 1d in Section 10.1 for a definition), but that is irrelevant to this question; the answer to the question asked is "yes."

---

26. Use an incidence matrix to represent the graphs in Exercises 1 and 2.

26. Each column represents an edge; the two 1's in the column are in the rows for the endpoints of the edge.

Exercise 1 $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Exercise 2 $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

---

27. Use an incidence matrix to represent the graphs in Exercises 13–15.

27. In an incidence matrix we have one column for each edge. We use alphabetical order of the vertices. Loops are represented by columns with one 1; other edges are represented by columns with two 1's. The order in which the columns are listed is immaterial.

Exercise 13 $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ Exercise 14 $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$

Exercise 15 $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$

***28.** What is the sum of the entries in a row of the adjacency matrix for an undirected graph? For a directed graph?

28. For an undirected graph, the sum of the entries in the i^{th} row is the same as the corresponding column sum, namely the number of edges incident to the vertex i, which is the same as the degree of i minus the number of loops at i (since each loop contributes 2 toward the degree count).

For a directed graph, the answer is dual to the answer for Exercise 29. The sum of the entries in the i^{th} row is the number of edges that have i as their initial vertex, i.e., the out-degree of i.

***29.** What is the sum of the entries in a column of the adjacency matrix for an undirected graph? For a directed graph?

29. In an undirected graph, each edge incident to a vertex j contributes 1 in the j^{th} column; thus the sum of the entries in that column is just the number of edges incident to j. Another way to state the answer is that the sum of the entries is the degree of j minus the number of loops at j, since each loop contributes 2 to the degree count.

In a directed graph, each edge whose terminal vertex is j contributes 1 in the j^{th} column; thus the sum of the entries in that column is just the number of edges that have j as their terminal vertex. Another way to state the answer is that the sum of the entries is the in-degree of j.

---

30. What is the sum of the entries in a row of the incidence matrix for an undirected graph?

30. The sum of the entries in the i^{th} row of the incidence matrix is the number of edges incident to vertex i, since there is one column with a 1 in row i for each such edge.

---

31. What is the sum of the entries in a column of the incidence matrix for an undirected graph?

31. Since each column represents an edge, the sum of the entries in the column is either 2, if the edge has 2 incident vertices (i.e., is not a loop), or 1 if it has only 1 incident vertex (i.e., is a loop).

***32.** Find an adjacency matrix for each of these graphs.

a) K_n **b)** C_n **c)** W_n **d)** $K_{m,n}$ **e)** Q_n

32. a) This is just the matrix that has 0's on the main diagonal and 1's elsewhere, namely

$$\begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

b) We label the vertices so that the cycle goes $v_1, v_2, \dots, v_n, v_1$. Then the matrix has 1's on the diagonals just above and below the main diagonal and in positions $(1, n)$ and $(n, 1)$, and 0's elsewhere:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

c) This matrix is the same as the answer in part (b), except that we add one row and column for the vertex

***33.** Find incidence matrices for the graphs in parts (a)–(d) of Exercise 32.

33. **a)** The incidence matrix for K_n has n rows and $C(n,2)$ columns. For each i and j with $1 \le i < j \le n$, there is a column with 1's in rows i and j and 0's elsewhere.

b) The matrix looks like this, with n rows and n columns.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

c) The matrix looks like the matrix for C_n, except with an extra row of 0's (which we have put at the end), since the vertex "in the middle" is not involved in the edges "around the outside," and n more columns for the "spokes." We show some extra space between the rim edge columns and the spoke columns; this is for human convenience only and does not have any bearing on the matrix itself.

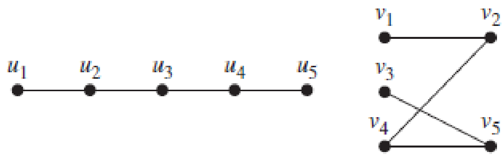
$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 & \quad 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & \quad 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & \quad 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & \quad 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \quad \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \quad 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 & \quad 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \quad 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

d) This matrix has $m+n$ rows and mn columns, one column for each pair (i,j) with $1 \le i \le m$ and $1 \le j \le n$. We have put in some extra spacing for readability of the pattern.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \quad 1 & 1 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \quad 0 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 1 & \cdots & 1 \\ & & & & & & & & & & & & \\ 1 & 0 & \cdots & 0 & \quad 1 & 0 & \cdots & 0 & \quad \cdots & \quad 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \quad 0 & 1 & \cdots & 0 & \quad \cdots & \quad 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \quad \vdots & \vdots & \ddots & \vdots & \quad \ddots & \quad \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \quad 0 & 0 & \cdots & 1 & \quad \cdots & \quad 0 & 0 & \cdots & 1 \end{bmatrix}$$

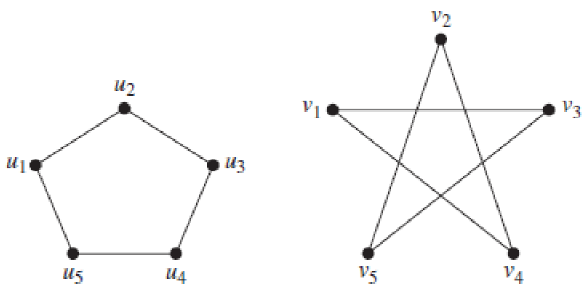
In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

34.



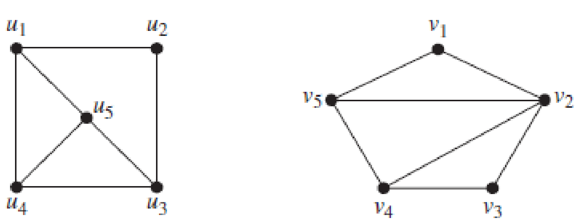
34. These graphs are isomorphic, since each is a path with five vertices. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_2$, $f(u_3) = v_4$, $f(u_4) = v_5$, and $f(u_5) = v_3$.

35.



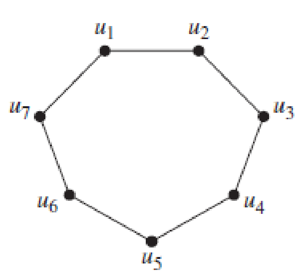
35. These graphs are isomorphic, since each is the 5-cycle. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_5$, $f(u_4) = v_2$, and $f(u_5) = v_4$.

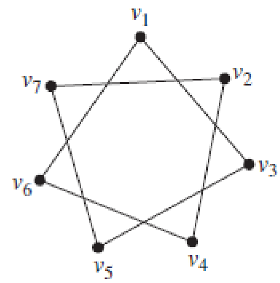
36.



36. These graphs are not isomorphic. The second has a vertex of degree 4, whereas the first does not.

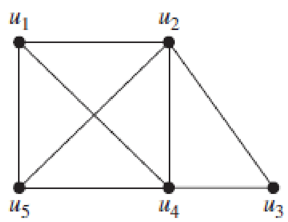
37.

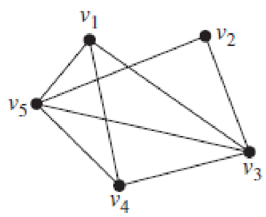




37. These graphs are isomorphic, since each is the 7-cycle (this is just like Exercise 35).

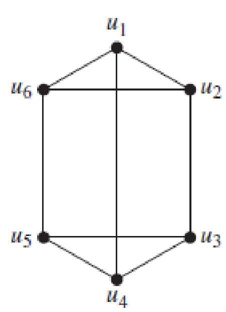
38.

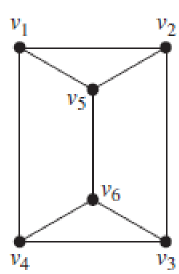




38. These two graphs are isomorphic. Each consists of a K_4 with a fifth vertex adjacent to two of the vertices in the K_4. Many isomorphisms are possible. One is $f(u_1) = v_1$, $f(u_2) = v_3$, $f(u_3) = v_2$, $f(u_4) = v_5$, and $f(u_5) = v_4$.

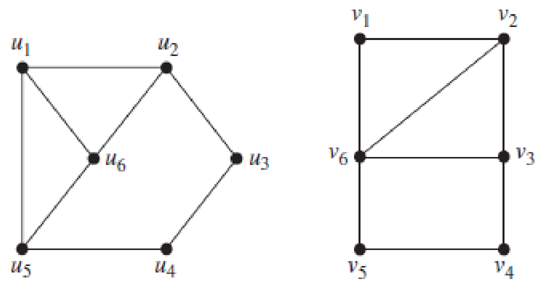
39.





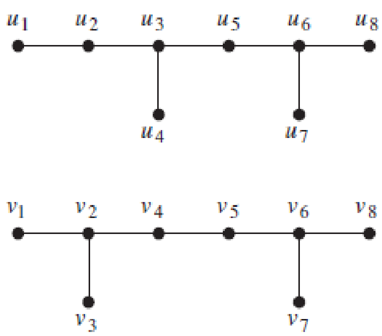
39. These two graphs are isomorphic. One can see this visually—just imagine "moving" vertices u_1 and u_4 into the inside of the rectangle, thereby obtaining the picture on the right. Formally, one isomorphism is $f(u_1) = v_5$, $f(u_2) = v_2$, $f(u_3) = v_3$, $f(u_4) = v_6$, $f(u_5) = v_4$, and $f(u_6) = v_1$.

40.



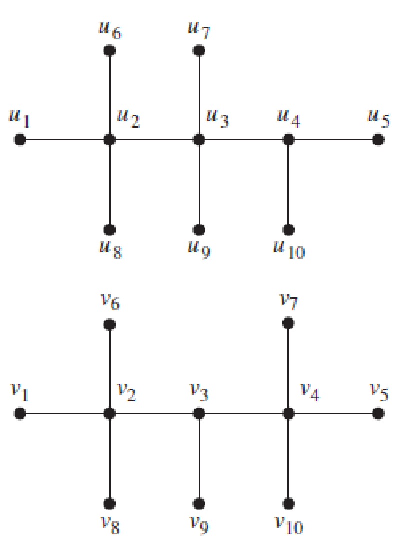
40. These graphs are not isomorphic—the degrees of the vertices are not the same (the graph on the right has a vertex of degree 4, which the graph on the left lacks).

41.



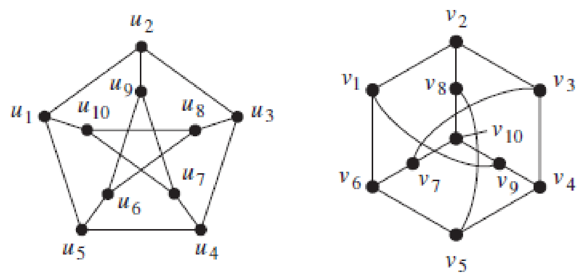
41. These graphs are not isomorphic. In the first graph the vertices of degree 3 are adjacent to a common vertex. This is not true of the second graph.

42.



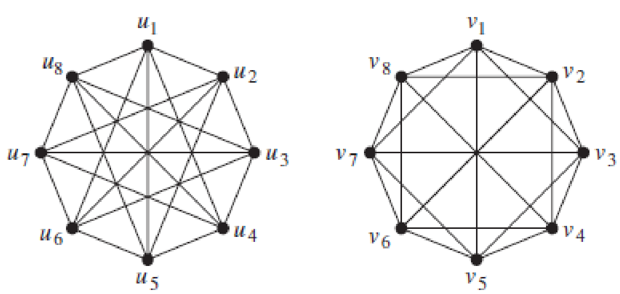
42. These graphs are not isomorphic. In the first graph the vertices of degree 4 are adjacent. This is not true of the second graph.

43.



43. These are isomorphic. One isomorphism is $f(u_1) = v_1$, $f(u_2) = v_9$, $f(u_3) = v_4$, $f(u_4) = v_3$, $f(u_5) = v_2$, $f(u_6) = v_8$, $f(u_7) = v_7$, $f(u_8) = v_5$, $f(u_9) = v_{10}$, and $f(u_{10}) = v_6$.

44.



44. The easiest way to show that these graphs are not isomorphic is to look at their complements. The complement of the graph on the left consists of two 4-cycles. The complement of the graph on the right is an 8-cycle. Since the complements are not isomorphic, the graphs are also not isomorphic.

---

45. Show that isomorphism of simple graphs is an equivalence relation.

45. We must show that being isomorphic is reflexive, symmetric, and transitive. It is reflexive since the identity function from a graph to itself provides the isomorphism (the one-to-one correspondence)—certainly the identity function preserves adjacency and nonadjacency. It is symmetric, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, then f^{-1} is a one-to-one correspondence that makes G_2 isomorphic to G_1; that is, f^{-1} is a one-to-one and onto function from V_2 to V_1 such that c and d are adjacent in G_2 if and only if $f^{-1}(c)$ and $f^{-1}(d)$ are adjacent in G_1. It is transitive, since if f is a one-to-one correspondence that makes G_1 isomorphic to G_2, and g is a one-to-one correspondence that makes G_2 isomorphic to G_3, then $g \circ f$ is a one-to-one correspondence that makes G_1 isomorphic to G_3.

46. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs $\overline{G}$ and $\overline{H}$ are also isomorphic.

46. This is immediate from the definition, since an edge is in $\overline{G}$ if and only if it is not in G, if and only if the corresponding edge is not in H, if and only if the corresponding edge is in $\overline{H}$.

---

47. Describe the row and column of an adjacency matrix of a graph corresponding to an isolated vertex.

47. If a vertex is isolated, then it has no adjacent vertices. Therefore in the adjacency matrix the row and column for that vertex must contain all 0's.

---

48. Describe the row of an incidence matrix of a graph corresponding to an isolated vertex.

48. An isolated vertex has no incident edges, so the row consists of all 0's.

---

49. Show that the vertices of a bipartite graph with two or more vertices can be ordered so that its adjacency matrix

has the form

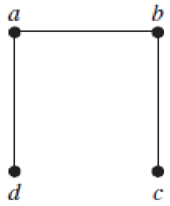
$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B} & \mathbf{0} \end{bmatrix},$$

where the four entries shown are rectangular blocks.

49. Let V_1 and V_2 be the two parts, say of sizes m and n, respectively. We can number the vertices so that all the vertices in V_1 come before all the vertices in V_2. The adjacency matrix has $m+n$ rows and $m+n$ columns. Since there are no edges between two vertices in V_1, the first m columns of the first m rows must all be 0's. Similarly, since there are no edges between two vertices in V_2, the last n columns of the last n rows must all be 0's. This is what we were asked to prove.

---

50. Show that this graph is self-complementary.



50. The complementary graph consists of edges $\{a, c\}$, $\{c, d\}$, and $\{d, b\}$; it is clearly isomorphic to the original graph (send d to a, a to c, b to d, and c to b).

---

51. Find a self-complementary simple graph with five vertices.

51. There are two such graphs, which can be found by trial and error. (We need only look for graphs with 5 vertices and 5 edges, since a self-complementary graph with 5 vertices must have $C(5,2)/2 = 5$ edges. If nothing else, we can draw them all and find the complement of each. See the pictures for the solution of Exercise 47d in Section 10.4.) One such graph is C_5. The other consists of a triangle, together with an edge from one vertex of the triangle to the fourth vertex, and an edge from another vertex of the triangle to the fifth vertex.

***52.** Show that if G is a self-complementary simple graph with v vertices, then $v \equiv 0$ or $1 \pmod 4$.

52. If G is self-complementary, then the number of edges of G must equal the number of edges of $\overline{G}$. But the sum of these two numbers is $n(n-1)/2$, where n is the number of vertices of G, since the union of the two graphs is K_n. Therefore the number of edges of G must be $n(n-1)/4$. Since this number must be an integer, a look at the four cases shows that n may be congruent to either 0 or 1, but not congruent to either 2 or 3, modulo 4.

---

53. For which integers n is C_n self-complementary?

53. If C_n is to be self-complementary, then C_n must have the same number of edges as its complement. We know that C_n has n edges. Its complement has the number of edges in K_n minus the number of edges in C_n, namely $C(n,2)-n = [n(n-1)/2]-n$. If we set these two quantities equal we obtain $[n(n-1)/2]-n = n$, which has $n = 5$ as its only solution. Thus C_5 is the only C_n that *might* be self-complementary—our argument just shows that it has the same number of edges as its complement, not that it is indeed isomorphic to its complement. However, it we draw C_5 and then draw its complement, then we see that the complement is again a copy of C_5. Thus $n = 5$ is the answer to the problem.

54. How many nonisomorphic simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

54. An excellent resource for questions of the form "how many nonisomorphic graphs are there with ...?" is Ronald C. Read and Robin J. Wilson, *An Atlas of Graphs* (Clarendon Press, 1998).

a) There are just two graphs with 2 vertices—the one with no edges, and the one with one edge.

b) A graph with three vertices can contain 0, 1, 2, or 3 edges. There is only one graph for each number of edges, up to isomorphism. Therefore the answer is 4.

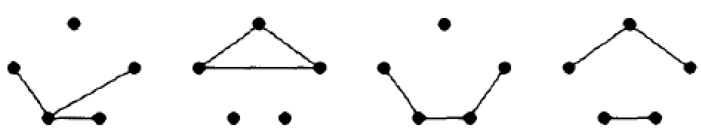
c) Here we look at graphs with 4 vertices. There is 1 graph with no edges, and 1 (up to isomorphism) with a single edge. If there are two edges, then these edges may or may not be adjacent, giving us 2 possibilities. If there are three edges, then the edges may form a triangle, a star, or a path, giving us 3 possibilities. Since graphs with four, five, or six edges are just complements of graphs with two, one, or no edges (respectively), the number of isomorphism classes must be the same as for these earlier cases. Thus our answer is $1 + 1 + 2 + 3 + 2 + 1 + 1 = 11$.

---

55. How many nonisomorphic simple graphs are there with five vertices and three edges?

55. We need to enumerate these graphs carefully to make sure of getting them all—leaving none out and not duplicating any. Let us organize our catalog by the degrees of the vertices. Since there are only 3 edges, the largest the degree could be is 3, and the only graph with 5 vertices, 3 edges, and a vertex of degree 3 is a $K_{1,3}$ together with an isolated vertex. If all the vertices that are not isolated have degree 2, then the graph must consist of a C_3 and 2 isolated vertices. The only way for there to be two vertices of degree 2 (and therefore also 2 of degree 1) is for the graph to be three edges strung end to end, together with an isolated vertex. The only other possibility is for 2 of the edges to be adjacent and the third to be not adjacent to either of the others. All in all, then, we have the 4 possibilities shown below.

See [ReWi] for more information about graph enumeration problems of this sort (such as Exercises 54, 56, and 68 in this section, Exercise 47 in Section 10.4, and supplementary exercises 2, 31, 32, and 40).



---

56. How many nonisomorphic simple graphs are there with six vertices and four edges?

56. There are 9 such graphs. Let us first look at the graphs that have a cycle in them. There is only 1 with a 4-cycle. There are 2 with a triangle, since the fourth edge can either be incident to the triangle or not. If there are no cycles, then the edges may all be in one connected component (see Section 10.4), in which case there are 3 possibilities (a path of length four, a path of length three with an edge incident to one of the middle vertices on the path, and a star). Otherwise, there are two components, which are necessarily either two paths of length two, a path of length three plus a single edge, or a star with three edges plus a single edge (3 possibilities in this case as well).

---

57. Are the simple graphs with the following adjacency matrices isomorphic?

a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

57. **a)** Both graphs consist of 2 sides of a triangle; they are clearly isomorphic.

b) The graphs are not isomorphic, since the first has 4 edges and the second has 5 edges.

c) The graphs are not isomorphic, since the first has 4 edges and the second has 3 edges.

58. Determine whether the graphs without loops with these incidence matrices are isomorphic.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

58. a) These graphs are both K_3, so they are isomorphic.

b) These are both simple graphs with 4 vertices and 5 edges. Up to isomorphism there is only one such graph (its complement is a single edge), so the graphs have to be isomorphic.

59. Extend the definition of isomorphism of simple graphs to undirected graphs containing loops and multiple edges.

59. There are at least two approaches we could take here. One approach is to have a correspondence not only of the vertices but also of the edges, with incidence (and nonincidence) preserved. In detail, we say that two pseudographs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there are one-to-one and onto functions $f : V_1 \to V_2$ and $g : E_1 \to E_2$ such that for each vertex $v \in V_1$ and edge $e \in E_1$, v is incident to e if and only if $f(v)$ is incident to $g(e)$.

Another approach is simply to count the number of edges between pairs of vertices. Thus we can define $G_1 = (V_1, E_1)$ to be isomorphic to $G_2 = (V_2, E_2)$ if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for every pair of (not necessarily distinct) vertices u and v in V_1, there are exactly the same number of edges in E_1 with $\{u, v\}$ as their set of endpoints as there are edges in E_2 with $\{f(u), f(v)\}$ as their set of endpoints.

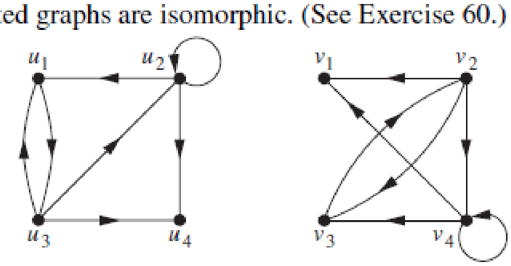
---

60. Define isomorphism of directed graphs.

60. We need only modify the definition of isomorphism of simple graphs slightly. The directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function $f : V_1 \to V_2$ such that for all pairs of vertices a and b in V_1, $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.

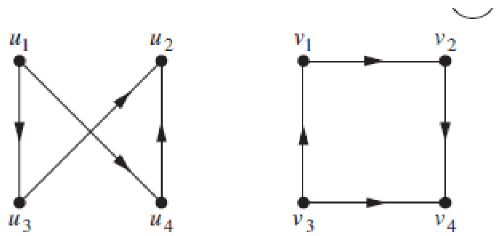
In Exercises 61–64 determine whether the given pair of directed graphs are isomorphic. (See Exercise 60.)

61.



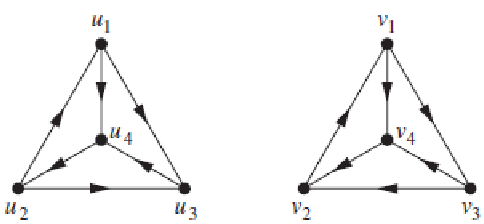
61. We can tell by looking at the loop, the parallel edges, and the degrees of the vertices that if these directed graphs are to be isomorphic, then the isomorphism has to be $f(u_1) = v_3$, $f(u_2) = v_4$, $f(u_3) = v_2$, and $f(u_4) = v_1$. We then need to check that each directed edge (u_i, u_j) corresponds to a directed edge $(f(u_i), f(u_j))$. We check that indeed it does for each of the 7 edges (and there are only 7 edges in the second graph). Therefore the two graphs are isomorphic.

62.



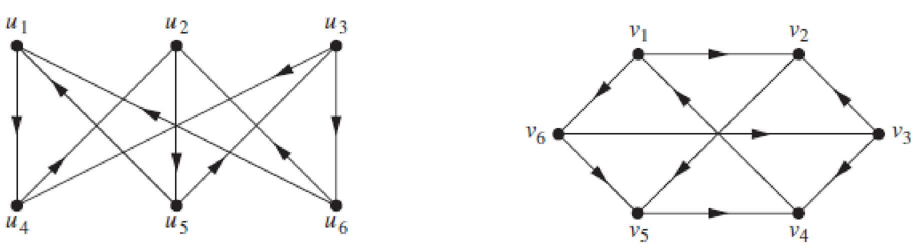
62. These two graphs are not isomorphic. In the first there is no edge from the unique vertex of in-degree 0 (u_1) to the unique vertex of out-degree 0 (u_2), whereas in the second graph there is such an edge, namely v_3v_4.

63.



63. If there is to be an isomorphism, the vertices with the same in-degree would have to correspond, and the edge between them would have to point in the same direction, so we would need u_1 to correspond to v_3, and u_2 to correspond to v_1. Similarly we would need u_3 to correspond to v_4, and u_4 to correspond to v_2. If we check all 6 edges under this correspondence, then we see that adjacencies are preserved (in the same direction), so the graphs are isomorphic.

64.



64. We claim that the digraphs are isomorphic. To discover an isomorphism, we first note that vertices u_1, u_2, and u_3 in the first digraph are independent (i.e., have no edges joining them), as are u_4, u_5, and u_6. Therefore these two groups of vertices will have to correspond to similar groups in the second digraph, namely v_1, v_3, and v_5, and v_2, v_4, and v_6, in some order. Furthermore, u_3 is the only vertex among one of these groups of u's to be the only one in the group with out-degree 2, so it must correspond to v_6, the vertex with the similar property in the other digraph; and in the same manner, u_4 must correspond to v_5. Now it is an easy matter, by looking at where the edges lead, to see that the isomorphism (if there is one) must also pair up u_1 with v_2; u_2 with v_4; u_5 with v_1; and u_6 with v_3. Finally, we easily verify that this indeed gives an isomorphism—each directed edge in the first digraph is present precisely when the corresponding directed edge is present in the second digraph.

***68.** How many nonisomorphic directed simple graphs are there with n vertices, when n is

a) 2? **b)** 3? **c)** 4?

68. a) There are 10 nonisomorphic directed graphs with 2 vertices. To see this, first consider graphs that have no edges from one vertex to the other. There are 3 such graphs, depending on whether they have no, one, or two loops. Similarly there are 3 in which there is an edge from each vertex to the other. Finally, there are 4 graphs that have exactly one edge between the vertices, because now the vertices are distinguished, and there can be or fail to be a loop at each vertex.

b) A detailed discussion of the number of directed graphs with 3 vertices would be rather long, so we will just give the answer, namely 104. There are some useful pictures relevant to this problem (and part (c) as well) in the appendix to *Graph Theory* by Frank Harary (Addison-Wesley, 1969).

c) The answer is 3069.

***69.** What is the product of the incidence matrix and its transpose for an undirected graph?

69. Suppose that the graph has v vertices and e edges. Then the incidence matrix is a $v \times e$ matrix, so its transpose is an $e \times v$ matrix. Therefore the product is a $v \times v$ matrix. Suppose that we denote the typical entry of this product by a_{ij}. Let t_{ik} be the typical entry of the incidence matrix; it is either a 0 or a 1. By definition

$$a_{ij} = \sum_{k=1}^{e} t_{ik}t_{jk}\,.$$

We can now read off the answer from this equation. If $i \neq j$, then a_{ij} is just a count of the number of edges incident to both i and j—in other words, the number of edges between i and j. On the other hand a_{ii} is equal to the number of edges incident to i.

***70.** How much storage is needed to represent a simple graph with n vertices and m edges using

a) adjacency lists?

b) an adjacency matrix?

c) an incidence matrix?

70. The answers depend on exactly how the storage is done, of course, but we will give naive answers that are at least correct as approximations.

a) We need one adjacency list for each vertex, and the list needs some sort of name or header; this requires n storage locations. In addition, each edge will appear twice, once in the list of each of its endpoints; this will require $2m$ storage locations. Therefore we need $n + 2m$ locations in all.

b) The adjacency matrix is a $n \times n$ matrix, so it requires n^2 bits of storage.

c) The incidence matrix is a $n \times m$ matrix, so it requires nm bits of storage.

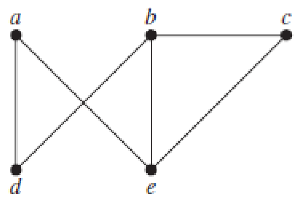
SECTION 10.4 Connectivity

Some of the most important uses of graphs deal with the notion of path, as the examples and exercises in this and subsequent sections show. It is important to understand the definitions, of course. Many of the exercises here are straightforward. The reader who wants to get a better feeling for what the arguments in more advanced graph theory are like should tackle problems like Exercises 35–38.

---

1. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, e, b, c, b **b)** a, e, a, d, b, c, a
c) e, b, a, d, b, e **d)** c, b, d, a, e, c



1. a) This is a path of length 4, but it is not simple, since edge $\{b, c\}$ is used twice. It is not a circuit, since it ends at a different vertex from the one at which it began.

b) This is not a path, since there is no edge from c to a.

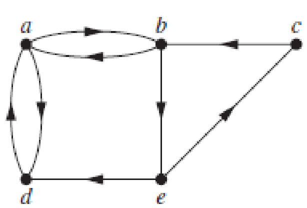
c) This is not a path, since there is no edge from b to a.

d) This is a path of length 5 (it has 5 edges in it). It is simple, since no edge is repeated. It is a circuit since it ends at the same vertex at which it began.

---

2. Does each of these lists of vertices form a path in the following graph? Which paths are simple? Which are circuits? What are the lengths of those that are paths?

a) a, b, e, c, b **b)** a, d, a, d, a
c) a, d, b, e, a **d)** a, b, e, c, b, d, a



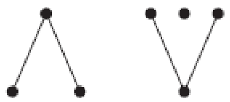
2. a) This is a path of length 4, but it is not a circuit, since it ends at a vertex other than the one at which it began. It is simple, since no edges are repeated.

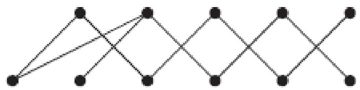
b) This is a path of length 4, which is a circuit. It is not simple, since it uses an edge more than once.

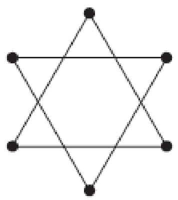
c) This is not a path, since there is no edge from d to b.

d) This is not a path, since there is no edge from b to d.

In Exercises 3–5 determine whether the given graph is connected.

3. 

4. 

5. 

3. This graph is not connected—it has three components.

4. This graph is connected—it is easy to see that there is a path from every vertex to every other vertex.

5. This graph is not connected. There is no path from the vertices in one of the triangles to the vertices in the other.

---

6. How many connected components does each of the graphs in Exercises 3–5 have? For each graph find each of its connected components.

6. The graph in Exercise 3 has three components: the piece that looks like a $\wedge$, the piece that looks like a $\vee$, and the isolated vertex. The graph in Exercise 4 is connected, with just one component. The graph in Exercise 5 has two components, each a triangle.

---

7. What do the connected components of acquaintanceship graphs represent?

7. A connected component of an acquaintanceship graph represent a maximal set of people with the property that for any two of them, we can find a string of acquaintances that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

---

8. What do the connected components of a collaboration graph represent?

8. A connected component of a collaboration graph represent a maximal set of people with the property that for any two of them, we can find a string of joint works that takes us from one to the other. The word "maximal" here implies that nobody else can be added to this set of people without destroying this property.

---

9. Explain why in the collaboration graph of mathematicians (see Example 3 in Section 10.1) a vertex representing a mathematician is in the same connected component as the vertex representing Paul Erdős if and only if that mathematician has a finite Erdős number.

9. If a person has Erdős number n, then there is a path of length n from that person to Erdős in the collaboration graph. By definition, that means that that person is in the same component as Erdős. Conversely, if a person is in the same component as Erdős, then there is a path from that person to Erdős, and the length of a shortest such path is that person's Erdős number.

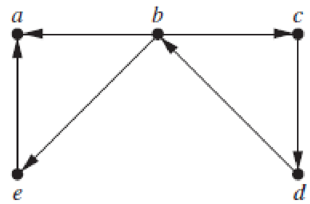
---

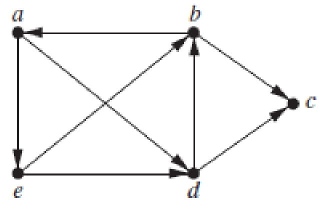
10. In the Hollywood graph (see Example 3 in Section 10.1), when is the vertex representing an actor in the same connected component as the vertex representing Kevin Bacon?

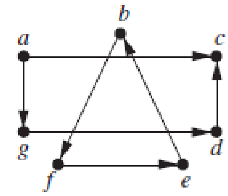
10. An actor is in the same connected component as Kevin Bacon if there is a path from that person to Bacon. This means that the actor was in a movie with someone who was in a movie with someone who . . . who was in a movie with Kevin Bacon. This includes Kevin Bacon, all actors who appeared in a movie with Kevin Bacon, all actors who appeared in movies with those people, and so on.

---

11. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

a)


b)


c)


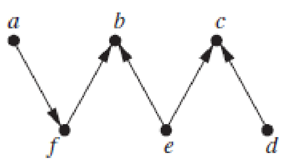
11. a) Notice that there is no path from a to any other vertex, because both edges involving a are directed toward a. Therefore the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

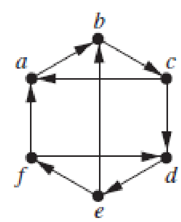
b) Notice that there is no path from c to any other vertex, because both edges involving c are directed toward c. Therefore the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

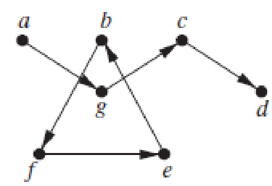
c) The underlying undirected graph is clearly not connected (one component has vertices b, f, and e), so this graph is neither strongly nor weakly connected.

---

12. Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

a)


b)


c)


12. a) Notice that there is no path from f to a, so the graph is not strongly connected. However, the underlying undirected graph is clearly connected, so this graph is weakly connected.

b) Notice that the sequence a, b, c, d, e, f, a provides a path from every vertex to every other vertex, so this graph is strongly connected.

c) The underlying undirected graph is clearly not connected (one component consists of the triangle), so this graph is neither strongly nor weakly connected.

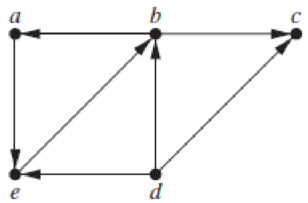
---

13. What do the strongly connected components of a telephone call graph represent?

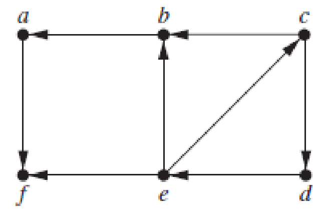
13. The strongly connected components are the maximal sets of phone numbers for which it is possible to find directed paths between every two different numbers in the set, where the existence of a directed path from phone number x to another phone number y means that x called some number, which called another number, $\ldots$, which called y. (The number of intermediary phone numbers in this path can be any natural number.)

14. Find the strongly connected components of each of these graphs.

a)



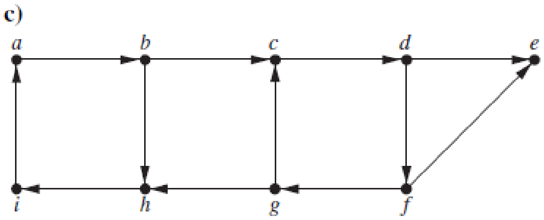
b)



a) The cycle $baeb$ guarantees that these three vertices are in one strongly connected component. Since there is no path from c to any other vertex, and there is no path from any other vertex to d, these two vertices are in strong components by themselves. Therefore the strongly connected components are $\{a, b, e\}$, $\{c\}$, and $\{d\}$.

b) The cycle $cdec$ guarantees that these three vertices are in one strongly connected component. The vertices a, b, and f are in strong components by themselves, since there are no paths both to and from each of these to every other vertex. Therefore the strongly connected components are $\{a\}$, $\{b\}$ $\{c, d, e\}$, and $\{f\}$.

c)

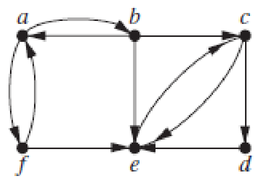


c) The cycle $abcdfghia$ guarantees that these eight vertices are in one strongly connected component. Since there is no path from e to any other vertex, this vertex is in a strong component by itself. Therefore the strongly connected components are $\{a, b, c, d, f, g, h, i\}$ and $\{e\}$.

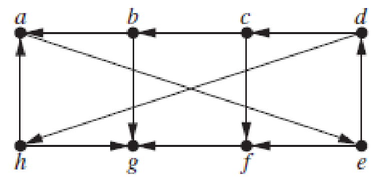
---

15. Find the strongly connected components of each of these graphs.

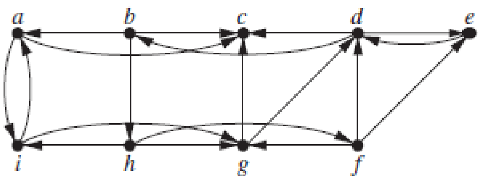
a)



b)



c)



15. In each case we want to look for large sets of vertices all which of which have paths to all the others. For these graphs, this can be done by inspection. These will be the strongly connected components.

a) Clearly $\{a, b, f\}$ is a set of vertices with paths between all the vertices in the set. The same can be said of $\{c, d, e\}$. Every edge between a vertex in the first set and a vertex in the second set is directed from the first, to the second. Hence there are no paths from c, d, or e to a, b, or f, and therefore these vertices are not in the same strongly connected component. Therefore these two sets are the strongly connected component.

b) The circuits a, e, d, c, b, a and a, e, d, h, a show that these six vertices are all in the same component. There is no path from f to any of these vertices, and no path from g to any other vertex. Therefore f and g are not in the same strong component as any other vertex. Therefore the strongly connected components are $\{a, b, c, d, e, h\}$, $\{f\}$, and $\{g\}$.

c) It is clear that a and i are in the same strongly connected component. If we look hard, we can also find the circuit b, h, f, g, d, e, d, b, so these vertices are in the same strongly connected component. Because of edges ig and hi, we can get from either of these collections to the other. Thus $\{a, b, d, e, f, g, h, i\}$ is a strong component. We cannot travel from c to any other vertex, so c is in a component by itself.

Suppose that $G = (V, E)$ is a directed graph. A vertex $w \in V$ is **reachable** from a vertex $v \in V$ if there is a directed path from v to w. The vertices v and w are **mutually reachable** if there are both a directed path from v to w and a directed path from w to v in G.

16. Show that if $G = (V, E)$ is a directed graph and u, v, and w are vertices in V for which u and v are mutually reachable and v and w are mutually reachable, then u and w are mutually reachable.

16. The given conditions imply that there is a path from u to v, a path from v to u, a path from v to w, and a path from w to v. Concatenating the first and third of these paths gives a path from u to w, and concatenating the fourth and second of these paths gives a path from w to u. Therefore u and w are mutually reachable.

---

17. Show that if $G = (V, E)$ is a directed graph, then the strong components of two vertices u and v of V are either the same or disjoint. [*Hint:* Use Exercise 16.]

17. The hardest part of this exercise is figuring out what we need to prove. It is enough to prove that if the strong components of u and v are not disjoint then they are the same. So suppose that w is a vertex that is in both the strong component of u and the strong component of v. (It is enough to consider the vertices in these components, because the edges in a strong component are just all the edges joining the vertices in that component.) This means that there are directed paths (in each direction) between u and w and between v and w. It follows that there are directed paths from u to v and from v to u, via w. Suppose x is a vertex in the strong component of u. Then x is also in the strong component of v, because there is a path from x to v (namely the path from x to u followed by the path from u to v) and vice versa.

---

18. Show that all vertices visited in a directed path connecting two vertices in the same strongly connected component of a directed graph are also in this strongly connected component.

18. Let $a, b, c, \ldots, z$ be the directed path. Since z and a are in the same strongly connected component, there is a directed path from z to a. This path appended to the given path gives us a circuit. We can reach any vertex on the original path from any other vertex on that path by going around this circuit.

---

19. Find the number of paths of length n between two different vertices in K_4 if n is

a) 2. **b)** 3. **c)** 4. **d)** 5.

19. One approach here is simply to invoke Theorem 2 and take successive powers of the adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The answers are the off-diagonal elements of these powers. An alternative approach is to argue combinatorially as follows. Without loss of generality, we assume that the vertices are called $1, 2, 3, 4$, and the path is to run from 1 to 2. A path of length n is determined by choosing the $n-1$ intermediate vertices. Each vertex in the path must differ from the one immediately preceding it.

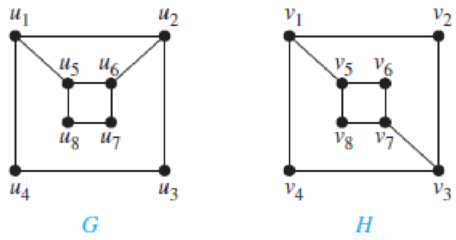
a) A path of length 2 requires the choice of 1 intermediate vertex, which must be different from both of the ends. Vertices 3 and 4 are the only ones available. Therefore the answer is 2.

b) Let the path be denoted $1, x, y, 2$. If $x = 2$, then there are 3 choices for y. If $x = 3$, then there are 2 choices for y; similarly if $x = 4$. Therefore there are $3 + 2 + 2 = 7$ possibilities in all.

c) Let the path be denoted $1, x, y, z, 2$. If $x = 3$, then by part **(b)** there are 7 choices for y and z. Similarly if $x = 4$. If $x = 2$, then y and z can be any two distinct members of $\{1, 3, 4\}$, and there are $P(3, 2) = 6$ ways to choose them. Therefore there are $7 + 7 + 6 = 20$ possibilities in all.

d) Let the path be denoted $1, w, x, y, z, 2$. If $w = 3$, then by part **(c)** there are 20 choices for x, y, and z. Similarly if $w = 4$. If $w = 2$, then x must be different from 2, and there are 3 choices for x. For each of these there are by part **(b)** 7 choices for y and z. This gives a total of 21 possibilities in this case. Therefore the answer is $20 + 20 + 21 = 61$.

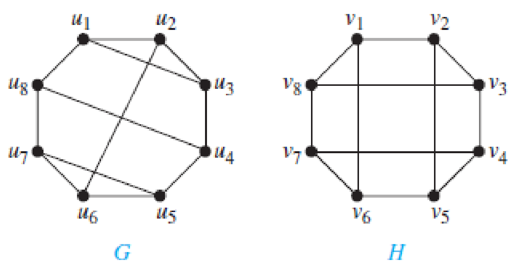
20. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.



20. The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

---

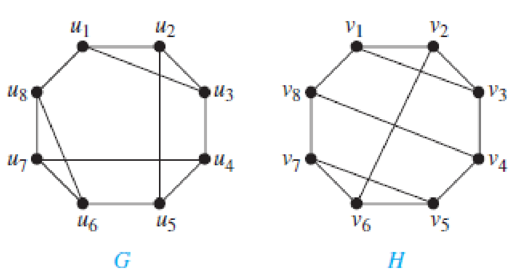
21. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



21. Graph G has a triangle (u_1, u_2, u_3). Graph H does not (in fact, it is bipartite). Therefore G and H are not isomorphic.

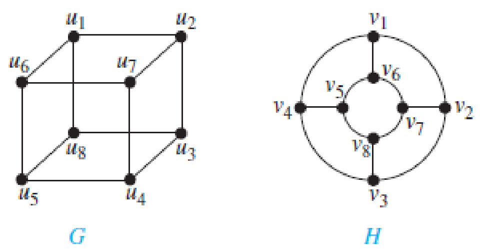
---

22. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



22. We notice that there are two vertices in each graph that are not in cycles of size 4. So let us try to construct an isomorphism that matches them, say $u_1 \leftrightarrow v_2$ and $u_8 \leftrightarrow v_6$. Now u_1 is adjacent to u_2 and u_3, and v_2 is adjacent to v_1 and v_3, so we try $u_2 \leftrightarrow v_1$ and $u_3 \leftrightarrow v_3$. Then since u_4 is the other vertex adjacent to u_3 and v_4 is the other vertex adjacent to v_3 (and we already matched u_3 and v_3), we must have $u_4 \leftrightarrow v_4$. Proceeding along similar lines, we then complete the bijection with $u_5 \leftrightarrow v_8$, $u_6 \leftrightarrow v_7$, and $u_7 \leftrightarrow v_5$. Having thus been led to the only possible isomorphism, we check that the 12 edges of G exactly correspond to the 12 edges of H, and we have proved that the two graphs are isomorphic.

23. Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



23. The drawing of G clearly shows it to be the cube Q_3. Can we see H as a cube as well? Yes—we can view the outer ring as the top face, and the inner ring as the bottom face. We can imagine walking around the top face of G clockwise (as viewed from above), then dropping down to the bottom face and walking around it counterclockwise, finally returning to the starting point on the top face. This is the path $u_1, u_2, u_7, u_6, u_5, u_4, u_3, u_8, u_1$. The corresponding path in H is $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$. We can verify that the edges not in the path do connect corresponding vertices. Therefore $G \cong H$.

---

24. Find the number of paths of length n between any two adjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

24. a) Adjacent vertices are in different parts, so every path between them must have odd length. Therefore there are no paths of length 2.

b) A path of length 3 is specified by choosing a vertex in one part for the second vertex in the path and a vertex in the other part for the third vertex in the path (the first and fourth vertices are the given adjacent vertices). Therefore there are $3 \cdot 3 = 9$ paths.

---

25. Find the number of paths of length n between any two nonadjacent vertices in $K_{3,3}$ for the values of n in Exercise 19.

25. As explained in the solution to Exercise 19, we could take powers of the adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The answers are found in location $(1, 2)$, for instance. Using the alternative approach is much easier than in Exercise 19. First of all, two nonadjacent vertices must lie in the same part, so only paths of even length can join them. Also, there are clearly 3 choices for each intermediate vertex in a path. Therefore we have the following answers:

a) $3^1 = 3$ **b)** 0 **c)** $3^3 = 27$ **d)** 0

26. Find the number of paths between c and d in the graph in Figure 1 of length

a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.

26. Probably the best way to do this is to write down the adjacency matrix for this graph and then compute its powers. The matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

a) To find the number of paths of length 2, we need to look at $\mathbf{A}^2$, which is

$$\begin{bmatrix} 3 & 1 & 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 3 & 2 & 2 \\ 2 & 1 & 3 & 0 & 3 & 1 \\ 1 & 3 & 0 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 & 4 & 1 \\ 2 & 2 & 1 & 2 & 1 & 3 \end{bmatrix}.$$

Since the $(3,4)^{\text{th}}$ entry is 0, so there are no paths of length 2.

b) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^3$ turns out to be 8, so there are 8 paths of length 3.

c) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^4$ turns out to be 10, so there are 10 paths of length 4.

d) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^5$ turns out to be 73, so there are 73 paths of length 5.

e) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^6$ turns out to be 160, so there are 160 paths of length 6.

f) The $(3,4)^{\text{th}}$ entry of $\mathbf{A}^7$ turns out to be 739, so there are 739 paths of length 7.

---

27. Find the number of paths from a to e in the directed graph in Exercise 2 of length

a) 2. b) 3. c) 4. d) 5. e) 6. f) 7.

27. There are two approaches here. We could use matrix multiplication on the adjacency matrix of this directed graph (by Theorem 2), which is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Thus we can compute $\mathbf{A}^2$ for part **(a)**, $\mathbf{A}^3$ for part **(b)**, and so on, and look at the $(1,5)^{\text{th}}$ entry to determine the number of paths from a to e. Alternately, we can argue in an ad hoc manner, as we do below.

a) There is just 1 path of length 2, namely a, b, e.

b) There are no paths of length 3, since after 3 steps, a path starting at a must be at b, c, or d.

c) For a path of length 4 to end at e, it must be at b after 3 steps. There are only 2 such paths, a, b, a, b, e and a, d, a, b, e.

d) The only way for a path of length 5 to end at e is for the path to go around the triangle bec. Therefore only the path a, b, e, c, b, e is possible.

e) There are several possibilities for a path of length 6. Since the only way to get to e is from b, we are asking for the number of paths of length 5 from a to b. We can go around the square (a, b, e, d, a, b), or else we can jog over to either b or d and back twice—there being 4 ways to choose where to do the jogging. Therefore there are 5 paths in all.

f) As in part **(d)**, it is clear that we have to use the triangle. We can either have a, b, a, b, e, c, b, e or a, d, a, b, e, c, b, e or a, b, e, c, b, a, b, e. Thus there are 3 paths.

***28.** Show that every connected graph with n vertices has at least $n-1$ edges.

28. We show this by induction on n. For $n=1$ there is nothing to prove. Now assume the inductive hypothesis, and let G be a connected graph with $n+1$ vertices and fewer than n edges, where $n \geq 1$. Since the sum of the degrees of the vertices of G is equal to 2 times the number of edges, we know that the sum of the degrees is less than $2n$, which is less than $2(n+1)$. Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1. Remove this vertex and its edge. Clearly the result is still connected, and it has n vertices and fewer than $n-1$ edges, contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

---

29. Let $G=(V,E)$ be a simple graph. Let R be the relation on V consisting of pairs of vertices (u,v) such that there is a path from u to v or such that $u=v$. Show that R is an equivalence relation.

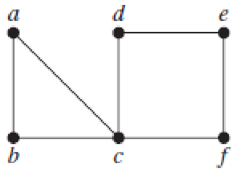
29. The definition given here makes it clear that u and v are related if and only if they are in the same component—in other words $f(u)=f(v)$ where $f(x)$ is the component in which x lies. Therefore by Exercise 9 in Section 9.5 this is an equivalence relation.

***30.** Show that in every simple graph there is a path from every vertex of odd degree to some other vertex of odd degree.

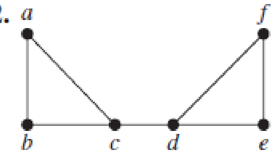
30. Let v be a vertex of odd degree, and let H be the component of G containing v. Then H is a graph itself, so it has an even number of vertices of odd degree. In particular, there is another vertex w in H with odd degree. By definition of connectivity, there is a path from v to w.

In Exercises 31–33 find all the cut vertices of the given graph.

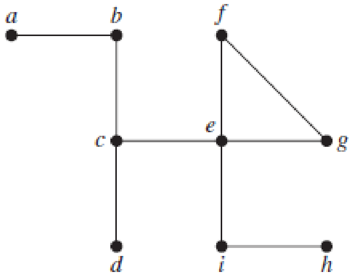
31.



32.



33.



31. A cut vertex is one whose removal splits the graph into more components than it originally had (which is 1 in this case). Only vertex c is a cut vertex here. If it is removed, then the resulting graph will have two components. If any other vertex is removed, then the graph remains connected.

32. Vertices c and d are the cut vertices. The removal of either one creates a graph with two components. The removal of any other vertex does not disconnect the graph.

33. There are several cut vertices here: b, c, e, and i. Removing any of these vertices creates a graph with more than one component. The removal of any of the other vertices leaves a graph with just one component.

---

34. Find all the cut edges in the graphs in Exercises 31–33.

34. The graph in Exercise 31 has no cut edges; any edge can be removed, and the result is still connected. For the graph in Exercise 32, $\{c,d\}$ is the only cut edge. There are several cut edges for the graph in Exercise 33: $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{c,e\}$, $\{e,i\}$, and $\{h,i\}$.

***35.** Suppose that v is an endpoint of a cut edge. Prove that v is a cut vertex if and only if this vertex is not pendant.

35. Without loss of generality, we can restrict our attention to the component in which the cut edge lies; other components of the graph are irrelevant to this proposition. To fix notation, let the cut edge be uv. When the cut edge is removed, the graph has two components, one of which contains v and the other of which contains u. If v is pendant, then it is clear that the removal of v results in exactly the component containing u—a connected graph. Therefore v is not a cut vertex in this case. On the other hand, if v is not pendant, then there are other vertices in the component containing v—at least one other vertex w adjacent to v. (We are assuming that this proposition refers to a simple graph, so that there is no loop at v.) Therefore when v is removed, there are at least two components, one containing u and another containing w.

***36.** Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

36. First we show that if c is a cut vertex, then there exist vertices u and v such that every path between them passes through c. Since the removal of c increases the number of components, there must be two vertices in G that are in different components after the removal of c. Then every path between these two vertices has to pass through c. Conversely, if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

***37.** Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices.

37. If every component of G is a single vertex, then clearly no vertex is a cut vertex (the removal of any of them actually decreases the number of components rather than increasing it). Therefore we may as well assume

---

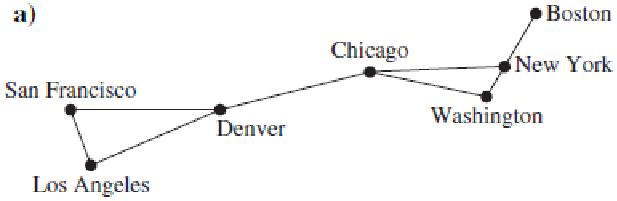
***38.** Show that an edge in a simple graph is a cut edge if and only if this edge is not part of any simple circuit in the graph.

38. First suppose that $e = \{u, v\}$ is a cut edge. Every circuit containing e must contain a path from u to v in addition to just the edge e. Since there are no such paths if e is removed from the graph, every such path must contain e. Thus e appears twice in the circuit, so the circuit is not simple. Conversely, suppose that e is not a cut edge. Then in the graph with e deleted u and v are still in the same component. Therefore there is a simple path P from u to v in this deleted graph. The circuit consisting of P followed by e is a simple circuit containing e.

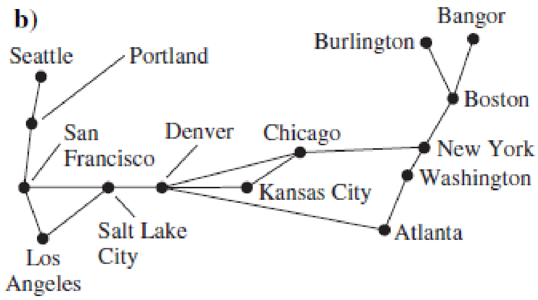
---

39. A communications link in a network should be provided with a backup link if its failure makes it impossible for some message to be sent. For each of the communications networks shown here in (a) and (b), determine those links that should be backed up.

a)



b)



39. This problem is simply asking for the cut edges of these graphs.

a) The link joining Denver and Chicago and the link joining Boston and New York are the cut edges.

b) The following links are the cut edges: Seattle–Portland, Portland–San Francisco, Salt Lake City–Denver, New York–Boston, Boston–Bangor, Boston–Burlington.

A **vertex basis** in a directed graph G is a minimal set B of vertices of G such that for each vertex v of G not in B there is a path to v from some vertex B.

40. Find a vertex basis for each of the directed graphs in Exercises 7–9 of Section 10.2.

40. In the directed graph in Exercise 7, there is a path from b to each of the other three vertices, so $\{b\}$ is a vertex basis (and a smallest one). It is easy to see that $\{c\}$ and $\{d\}$ are also vertex bases, but a is not in any vertex basis. For the directed graph in Exercise 8, there is a path from b to each of a and c; on the other hand, d must clearly be in every vertex basis. Thus $\{b, d\}$ is a smallest vertex basis. So are $\{a, d\}$ and $\{c, d\}$. Every vertex basis for the directed graph in Exercise 9 must contain vertex e, since it has no incoming edges. On the other hand, from any other vertex we can reach all the other vertices, so e together with any one of the other four vertices will form a vertex basis.

---

41. What is the significance of a vertex basis in an influence graph (described in Example 2 of Section 10.1)? Find a vertex basis in the influence graph in that example.

41. A vertex basis will be a set of people who collectively can influence everyone, at least indirectly, but none of whom influences another member of that set (otherwise the set would not be minimal). The set consisting of Deborah is a vertex basis, since she can influence everyone except Yvonne directly, and she can influence Yvonne indirectly through Brian.

---

42. Show that if a connected simple graph G is the union of the graphs G_1 and G_2, then G_1 and G_2 have at least one common vertex.

42. By definition of graph, both G_1 and G_2 are nonempty. If they have no common vertex, then there clearly can be no paths from $v_1 \in G_1$ to $v_2 \in G_2$. In that case G would not be connected, contradicting the hypothesis.

***43.** Show that if a simple graph G has k connected components and these components have $n_1, n_2, \ldots, n_k$ vertices, respectively, then the number of edges of G does not exceed

$$\sum_{i=1}^{k} C(n_i, 2).$$

43. Since there can be no edges between vertices in different components, G will have the most edges when each of the components is a complete graph. Since K_{n_i} has $C(n_i, 2)$ edges, the maximum number of edges is the sum given in the exercise.

***44.** Use Exercise 43 to show that a simple graph with n vertices and k connected components has at most $(n-k)(n-k+1)/2$ edges. [*Hint:* First show that

$$\sum_{i=1}^{k} n_i^2 \le n^2 - (k-1)(2n-k),$$

where n_i is the number of vertices in the ith connected component.]

44. First we obtain the inequality given in the hint. We claim that the maximum value of $\sum n_i^2$, subject to the constraint that $\sum n_i = n$, is obtained when one of the n_i's is as large as possible, namely $n-k+1$, and the remaining n_i's (there are $k-1$ of them) are all equal to 1. To justify this claim, suppose instead that two of the n_i's were a and b, with $a \ge b \ge 2$. If we replace a by $a+1$ and b by $b-1$, then the constraint is still satisfied, and the sum of the squares has changed by $(a+1)^2+(b-1)^2-a^2-b^2 = 2(a-b)+2 \ge 2$. Therefore the maximum cannot be attained unless the n_i's are as we claimed. Since there are only a finite number of possibilities for the distribution of the n_i's, the arrangement we give must in fact yield the maximum. Therefore $\sum n_i^2 \le (n-k+1)^2 + (k-1)\cdot 1^2 = n^2-(k-1)(2n-k)$, as desired.

Now by Exercise 43, the number of edges of the given graph does not exceed $\sum C(n_i,2) = \sum (n_i^2+n_i)/2 = ((\sum n_i^2)+n)/2$. Applying the inequality obtained above, we see that this does not exceed $(n^2-(k-1)(2n-k)+n)/2$, which after a little algebra is seen to equal $(n-k)(n-k+1)/2$. The upshot of all this is that the most edges are obtained if there is one component as large as possible, with all the other components consisting of isolated vertices.

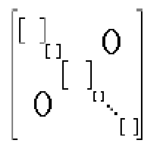
***45.** Show that a simple graph G with n vertices is connected if it has more than $(n-1)(n-2)/2$ edges.

45. Before we give a correct proof here, let us look at an incorrect proof that students often give for this exercise. It goes something like this. "Suppose that the graph is not connected. Then no vertex can be adjacent to every other vertex, only to $n-2$ other vertices. One vertex joined to $n-2$ other vertices creates a component with $n-1$ vertices in it. To get the most edges possible, we must use all the edges in this component. The number of edges in this component is thus $C(n-1,2) = (n-1)(n-2)/2$, and the other component (with only one vertex) has no edges. Thus we have shown that a disconnected graph has at most $(n-1)(n-2)/2$ edges, so every graph with more edges than that has to be connected." The fallacy here is in assuming—without justification—that the maximum number of edges is achieved when one component has $n-1$ vertices. What if, say, there were two components of roughly equal size? Might they not together contain more edges? We will see that the answer is "no," but it is important to realize that this requires proof—it is not obvious without some calculations.

Here is a correct proof, then. Suppose that the graph is not connected. Then it has a component with k vertices in it, for some k between 1 and $n-1$, inclusive. The remaining $n-k$ vertices are in one or more other components. The maximum number of edges this graph could have is then $C(k,2)+C(n-k,2)$, which, after a bit of algebra, simplifies to $k^2-nk+(n^2-n)/2$. This is a quadratic function of k. It is minimized when $k=n/2$ (the k coordinate of the vertex of the parabola that is the graph of this function) and maximized at the endpoints of the domain, namely $k=1$ and $k=n-1$. In the latter cases its value is $(n-1)(n-2)/2$. Therefore the largest number of edges that a disconnected graph can have is $(n-1)(n-2)/2$, so every graph with more edges than this must be connected.

46. Describe the adjacency matrix of a graph with n connected components when the vertices of the graph are listed so that vertices in each connected component are listed successively.

46. Under these conditions, the matrix has a block structure, with all the 1's confined to small squares (of various sizes) along the main diagonal. The reason for this is that there are no edges between different components. See the picture for a schematic view. The only 1's occur inside the small submatrices (but not all the entries in these squares are 1's, of course).



---

48. Show that each of the following graphs has no cut vertices.

a) C_n where $n \geq 3$
b) W_n where $n \geq 3$
c) $K_{m,n}$ where $m \geq 2$ and $n \geq 2$
d) Q_n where $n \geq 2$

48. a) If any vertex is removed from C_n, the graph that remains is a connected graph, namely a path with $n-1$ vertices.

b) If the central vertex is removed, the resulting graph is a cycle, which is connected. If a vertex on the cycle of W_n is removed, the resulting graph is connected because every remaining vertex on the cycle is joined to the central vertex.

c) Let v be a vertex in one part and w a vertex in the other part, after some vertex has been removed (these exists because m and n are both greater than 1). Then v and w are joined by an edge, and every other vertex is joined by an edge to either v or w, giving us a connected graph.

d) We can use mathematical induction, based on the recursive definition of the n-cubes (see Example 8 in Section 10.2). The basis step is Q_2, which is the same as C_4, and we argued in part (a) that it has no cut vertex. Assume the inductive hypothesis. Let G be Q_{k+1} with a vertex removed. Then G consists of a copy of Q_k, which is certainly connected, a copy of Q_k with a vertex removed, which is connected by the inductive hypothesis, and at least one edge joining those two subgraphs; therefore G is connected.

47. How many nonisomorphic connected simple graphs are there with n vertices when n is

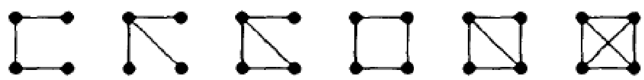
a) 2? **b)** 3? **c)** 4? **d)** 5?

47. We have to enumerate carefully all the possibilities.

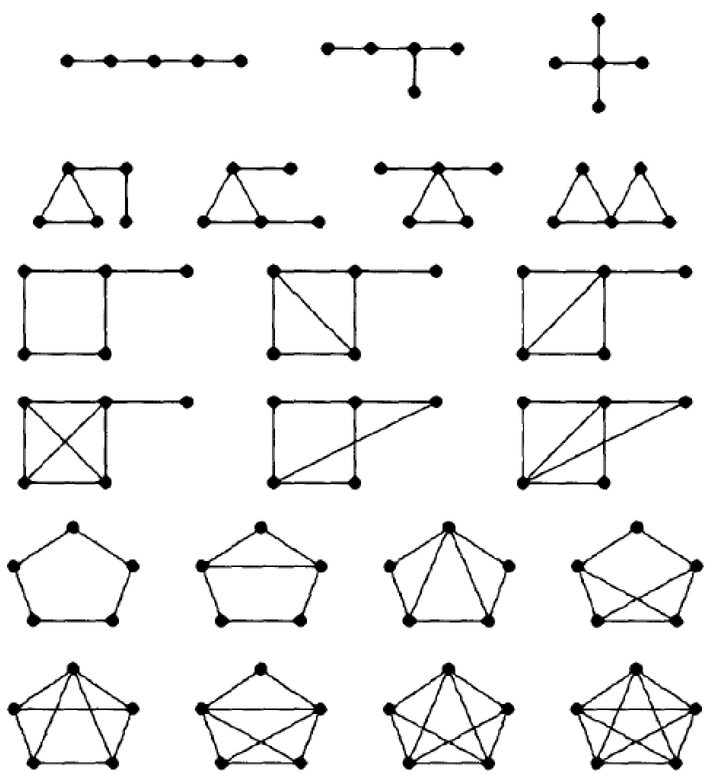
a) There is obviously only 1, namely K_2, the graph consisting of two vertices and the edge between them.

b) There are clearly 2 connected graphs with 3 vertices, namely K_3 and K_3 with one edge deleted, as shown.

c) There are several connected graphs with $n = 4$. If the graph has no circuits, then it must either be a path of length 3 or the "star" $K_{1,3}$. If it contains a triangle but no copy of C_4, then the other vertex must be pendant—only 1 possibility. If it contains a copy of C_4, then neither, one, or both of the other two edges may be present—3 possibilities. Therefore the answer is $2 + 1 + 3 = 6$. The graphs are shown below.



d) We need to enumerate the possibilities in some systematic way, such as by the largest cycle contained in the graph. There are 21 such graphs, as can be seen by such an enumeration, shown below. First we show those graphs with no circuits, then those with a triangle but no C_4 or C_5, then those with a C_4 but no C_5, and finally those with a C_5. In doing this problem we have to be careful not only not to leave out any graphs, but also not to list any twice.



49. Show that each of the graphs in Exercise 48 has no cut edges.

49. In each case we just need to verify that the removal of an edge will not disconnect the graph.

a) Removing an edge from a cycle leaves a path, which is still connected.

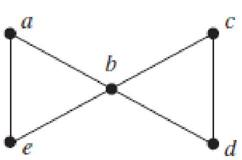
b) Removing an edge from the cycle portion of the wheel leaves that portion still connected as in part **(a)**, and the central vertex is clearly still connected to it as well. Removing a spoke leaves the cycle intact and the central vertex still connected to it as well.

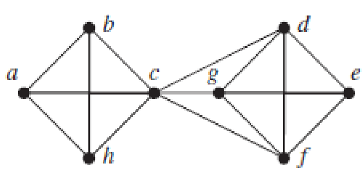
c) Let u, v, a, b be any four vertices of $K_{m,n}$ with u and v in one part and a and b in the other. They are connected by the 4-cycle $uavb$. Removing one edge will not disconnect this 4-cycle, so these vertices are still connected, and the entire graph is therefore still connected. Note that we needed $m, n \geq 2$ for this to work (and for the statement to be true).

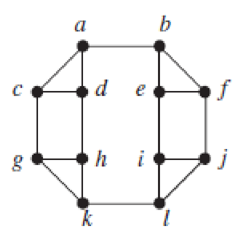
d) Think of Q_n as two copies of Q_{n-1} with corresponding vertices joined by an edge. Without loss of generality we can assume that the removed edge is one of the edges joining corresponding vertices. Since each Q_{n-1} is connected and at least one edge remains joining the two copies, the resulting graph is connected.

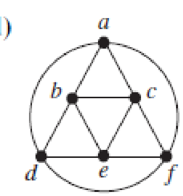
---

50. For each of these graphs, find $\kappa(G)$, $\lambda(G)$, and $\min_{v\in V} \deg(v)$, and determine which of the two inequalities in $\kappa(G) \leq \lambda(G) \leq \min_{v\in V} \deg(v)$ are strict.

a)


b)


c)


d)


50. a) Removing vertex b leaves two components, so $\kappa(G) = 1$. Removing one edge does not disconnect the graph, but removing edges ab and eb do disconnect the graph, so $\lambda(G) = 2$. The minimum degree is clearly 2. Thus only $\kappa(G) < \lambda(G)$ is strict.

b) Removing vertex c leaves two components, so $\kappa(G) = 1$. It is not hard to see that removing two edges does not disconnect the graph, but removing the three edges incident to vertex a, for example, does. Therefore $\lambda(G) = 3$. Since the minimum degree is also 3, only $\kappa(G) < \lambda(G)$ is a strict inequality.

c) It is easy to see that removing only one vertex or one edge does not disconnect this graph, but removing vertices a and k, or removing edges ab and kl, does. Therefore $\kappa(G) = \lambda(G) = 2$. Since the minimum degree is 3, only the inequality $\lambda(G) < \min_{v\in V} \deg(v)$ is strict.

d) With a little effort we see that $\kappa(G) = \lambda(G) = \min_{v\in V} \deg(v) = 4$, so none of the inequalities is strict.

51. Show that if G is a connected graph, then it is possible to remove vertices to disconnect G if and only if G is not a complete graph.

51. If G is complete, then removing vertices one by one leaves a complete graph at each step, so we never get a disconnected graph. Conversely, if G is not complete, say with edge uv missing, then removing all the vertices except u and v creates the disconnected graph consisting of just those two vertices.

---

52. Show that if G is a connected graph with n vertices then

a) $\kappa(G) = n - 1$ if and only if $G = K_n$.

b) $\lambda(G) = n - 1$ if and only if $G = K_n$.

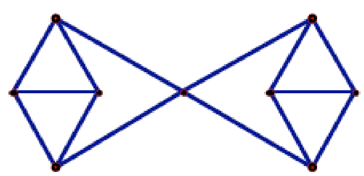
52. a) According to the discussion following Example 7, $\kappa(K_n) = n - 1$. Conversely, if G is a graph with n vertices other than K_n, let u and v be two nonadjacent vertices of G. Then removing the $n-2$ vertices other than u and v disconnects G, so $\kappa(G) < n - 1$.

b) Since $\kappa(K_n) \leq \lambda(K_n) \leq \min_{v \in K_n} \deg(v)$ (see the discussion following Example 9) and the outside quantities are both $n - 1$, it follows that $\lambda(K_n) = n - 1$. Conversely, if G is not K_n, then its minimum degree is less than $n - 1$, so it edge connectivity is also less than $n - 1$.

---

54. Construct a graph G with $\kappa(G) = 1$, $\lambda(G) = 2$, and $\min_{v \in V} \deg(v) = 3$.

54. Here is one example.



---

57. Use Theorem 2 to find the length of the shortest path between a and f in the graph in Figure 1.

57. We need to look at successive powers of the adjacency matrix until we find one in which the $(1,6)^{\text{th}}$ entry is not 0. Since the matrix is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

we see that the $(1,6)^{\text{th}}$ entry of $\mathbf{A}^2$ is 2. Thus there is a path of length 2 from a to f (in fact 2 of them). On the other hand there is no path of length 1 from a to f (i.e., no edge), so the length of a shortest path is 2.

---

58. Use Theorem 2 to find the length of the shortest path from a to c in the directed graph in Exercise 2.

58. First we write down the adjacency matrix for this graph, namely

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then we compute A^2 and A^3, and look at the $(1,3)^{\text{th}}$ entry of each. We find that these entries are 0 and 1, respectively. By the reasoning given in Exercise 57, we conclude that a shortest path has length 3.

---

59. Let P_1 and P_2 be two simple paths between the vertices u and v in the simple graph G that do not contain the same set of edges. Show that there is a simple circuit in G.

59. Let the simple paths P_1 and P_2 be $u = x_0, x_1, \ldots, x_n = v$ and $u = y_0, y_1, \ldots, y_m = v$, respectively. The paths thus start out at the same vertex. Since the paths do not contain the same set of edges, they must diverge eventually. If they diverge only after one of them has ended, then the rest of the other path is a simple circuit from v to v. Otherwise we can suppose that $x_0 = y_0$, $x_1 = y_1$, ..., $x_i = y_i$, but $x_{i+1} \neq y_{i+1}$. To form our simple circuit, we follow the path y_i, y_{i+1}, y_{i+2}, and so on, until it once again first encounters a vertex on P_1 (possibly as early as y_{i+1}, no later than y_m). Once we are back on P_1, we follow it along—forwards or backwards, as necessary—to return to x_i. Since $x_i = y_i$, this certainly forms a circuit. It must be a simple circuit, since no edge among the x_k's or the y_l's can be repeated (P_1 and P_2 are simple by hypothesis) and no edge among the x_k's can equal one of the edges y_l that we used, since we abandoned P_2 for P_1 as soon as we hit P_1.

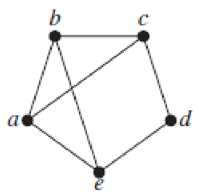
SECTION 10.5 Euler and Hamilton Paths

An Euler circuit or Euler path uses every edge exactly once. A Hamilton circuit or Hamilton path uses every vertex exactly once (not counting the circuit's return to its starting vertex). Euler and Hamilton circuits and paths have an important place in the history of graph theory, and as we see in this section they have some interesting applications. They provide a nice contrast—there are good algorithms for finding Euler paths (see also Exercises 50–53), but computer scientists believe that there is no good (efficient) algorithm for finding Hamilton paths.

Most of these exercises are straightforward. The reader should at least look at Exercises 16 and 17 to see how the concept of Euler path applies to directed graphs—these exercises are not hard if you understood the proof of Theorem 1 (given in the text before the statement of the theorem).

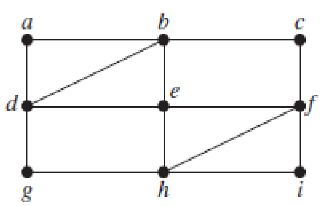
In Exercises 1–8 determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

1.

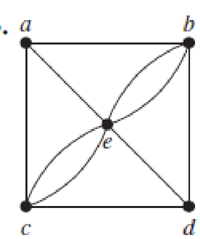


1. Since there are four vertices of odd degree (a, b, c, and e) and $4 > 2$, this graph has neither an Euler circuit nor an Euler path.

2.

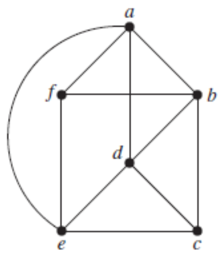


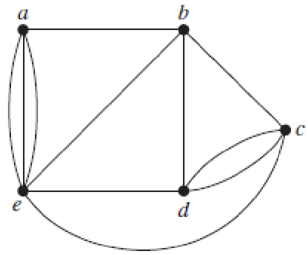
3.



2. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, f, i, h, g, d, e, h, f, e, b, d, a$.

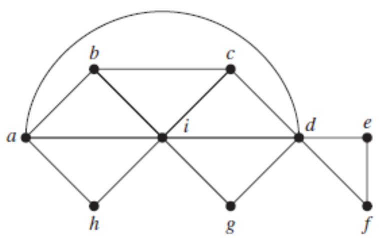
3. Since there are two vertices of odd degree (a and d), this graph has no Euler circuit, but it does have an Euler path starting at a and ending at d. We can find such a path by inspection, or by using the splicing idea explained in this section. One such path is $a, e, c, e, b, e, d, b, a, c, d$.

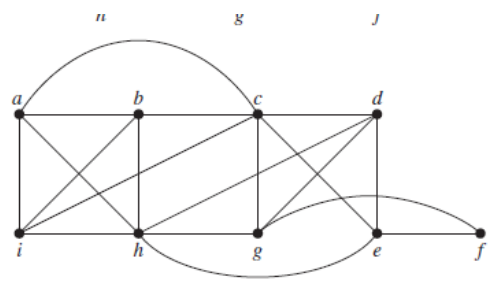
4.


5.


4. This graph has no Euler circuit, since the degree of vertex c (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $f, a, b, c, d, e, f, b, d, a, e, c$.

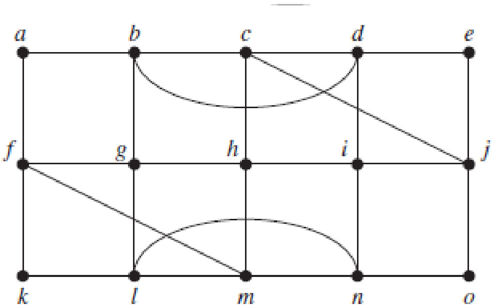
5. All the vertex degrees are even, so there is an Euler circuit. We can find such a circuit by inspection, or by using the splicing idea explained in this section. One such circuit is $a, b, c, d, c, e, d, b, e, a, e, a$.

6.


7.


6. This graph has no Euler circuit, since the degree of vertex b (for one) is odd. There is an Euler path between the two vertices of odd degree. One such path is $b, c, d, e, f, d, g, i, d, a, h, i, a, b, i, c$.

7. All the vertex degrees are even, so there is an Euler circuit. We can find such a circuit by inspection, or by using the splicing idea explained in this section. One such circuit is $a, b, c, d, e, f, g, h, i, a, h, b, i, c, e, h, d, g, c, a$.

8.


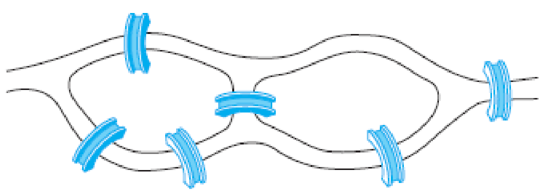
8. All the vertex degrees are even, so there is an Euler circuit. We can find one by trial and error, or by using Algorithm 1. One such circuit is $a, b, c, d, e, j, c, h, i, d, b, g, h, m, n, o, j, i, n, l, m, f, g, l, k, f, a$.

9. Suppose that in addition to the seven bridges of Königsberg (shown in Figure 1) there were two additional bridges, connecting regions B and C and regions B and D, respectively. Could someone cross all nine of these bridges exactly once and return to the starting point?

9. No, an Euler circuit does not exist in the graph modeling this hypothetical city either. Vertices A and B have odd degree.

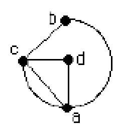
---

10. Can someone cross all the bridges shown in this map exactly once and return to the starting point?



2

10. The graph model for this exercise is as shown here.



Vertices a and b are the banks of the river, and vertices c and d are the islands. Each vertex has even degree, so the graph has an Euler circuit, such as a, c, b, a, d, c, a. Therefore a walk of the type described is possible.

---

11. When can the centerlines of the streets in a city be painted without traveling a street more than once? (Assume that all the streets are two-way streets.)

11. Assuming we have just one truck to do the painting, the truck must follow an Euler path through the streets in order to do the job without traveling a street twice. Therefore this can be done precisely when there is an Euler path or circuit in the graph, which means that either zero or two vertices (intersections) have odd degree (number of streets meeting there). We are assuming, of course, that the city is connected.

---

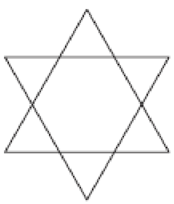
12. Devise a procedure, similar to Algorithm 1, for constructing Euler paths in multigraphs.

In Exercises 13–15 determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

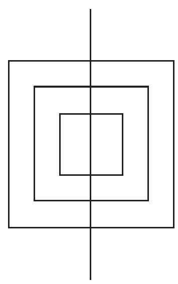
12. The algorithm is essentially the same as Algorithm 1. If there are no vertices of odd degree, then we simply use Algorithm 1, of course. If there are exactly two vertices of odd degree, then we begin constructing the initial path at one such vertex, and it will necessarily end at the other when it cannot be extended any further. Thereafter we follow Algorithm 1 exactly, splicing new circuits into the path we have constructed so far until no unused edges remain.

In Exercises 13–15 determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

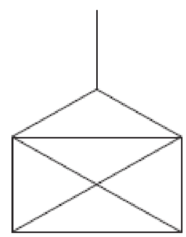
13.



14.



15.



13. In order for the picture to be drawn under the conditions of Exercises 13–15, the graph formed by the picture must have an Euler path or Euler circuit. Note that all of these graphs are connected. The graph in the current exercise has all vertices of even degree; therefore it has an Euler circuit and can be so traced.

14. See the comments in the solution to Exercise 13. This graph has exactly two vertices of odd degree; therefore it has an Euler path and can be so traced.

15. See the comments in the solution to Exercise 13. This graph has 4 vertices of odd degree; therefore it has no Euler path or circuit and cannot be so traced.

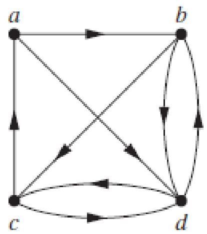
***16.** Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

16. First suppose that the directed multigraph has an Euler circuit. Since this circuit provides a path from every vertex to every other vertex, the graph must be strongly connected (and hence also weakly connected). Also, we can count the in-degrees and out-degrees of the vertices by following this circuit; as the circuit passes through a vertex, it adds one to the count of both the in-degree (as it comes in) and the out-degree (as it leaves). Therefore the two degrees are equal for each vertex.

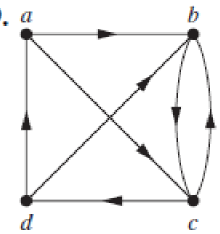
Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem 1 and construct an Euler circuit.

In Exercises 18–23 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.

18.



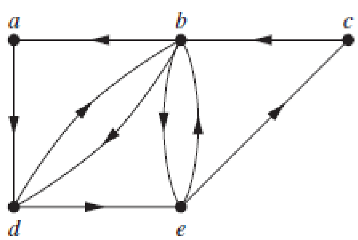
19.



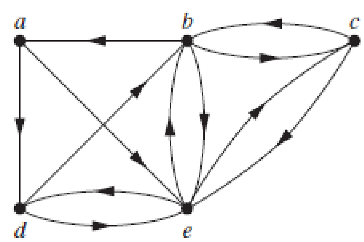
18. For Exercises 18–23 we use the results of Exercises 16 and 17. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from a to d. One such path is $a, b, d, b, c, d, c, a, d$.

19. For Exercises 18–23 we use the results of Exercises 16 and 17. By Exercise 16, we cannot hope to find an Euler circuit since vertex b has different out-degree and in-degree. By Exercise 17, we cannot hope to find an Euler path since vertex b has out-degree and in-degree differing by 2.

20.



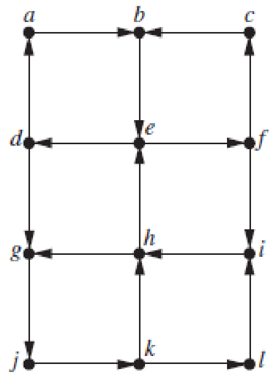
21.



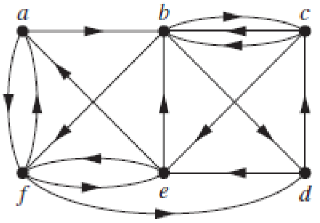
20. The conditions of Exercise 16 are met, so there is an Euler circuit, which is perforce also an Euler path. One such path is $a, d, b, d, e, b, e, c, b, a$.

21. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from a to e. One such path is $a, d, e, d, b, a, e, c, e, b, c, b, e$.

23.



22.



22. This directed graph satisfies the condition of Exercise 17 but not that of Exercise 16. Therefore there is no Euler circuit. The Euler path must go from c to b. One such path is $c, e, b, d, c, b, f, d, e, f, e, a, f, a, b, c, b$. (There is no Euler circuit, however, since the conditions of Exercise 16 are not met.)

23. There are more than two vertices whose in-degree and out-degree differ by 1, so by Exercises 16 and 17, there is no Euler path or Euler circuit.

***24.** Devise an algorithm for constructing Euler circuits in directed graphs.

24. The algorithm is identical to Algorithm 1.

---

25. Devise an algorithm for constructing Euler paths in directed graphs.

25. The algorithm is very similar to Algorithm 1. The input is a weakly connected directed multigraph in which either each vertex has in-degree equal to its out-degree, or else all vertices except two satisfy this condition and the remaining vertices have in-degree differing from out-degree by 1 (necessarily once in each direction). We begin by forming a path starting at the vertex whose out-degree exceeds its in-degree by 1 (in the second case) or at any vertex (in the first case). We traverse the edges (never more than once each), forming a path, until we cannot go on. Necessarily we end up either at the vertex whose in-degree exceeds its out-degree (in the first case) or at the starting vertex (in the second case). From then on we do exactly as in Algorithm 1, finding a simple circuit among the edges not yet used, starting at any vertex on the path we already have; such a vertex exists by the weak connectivity assumption. We splice this circuit into the path, and repeat the process until all edges have been used.

---

26. For which values of n do these graphs have an Euler circuit?

a) K_n **b)** C_n **c)** W_n **d)** Q_n

26. a) The degrees of the vertices $(n-1)$ are even if and only if n is odd. Therefore there is an Euler circuit if and only if n is odd (and greater than 1, of course).

b) For all $n \geq 3$, clearly C_n has an Euler circuit, namely itself.

c) Since the degrees of the vertices around the rim are all odd, no wheel has an Euler circuit.

d) The degrees of the vertices are all n. Therefore there is an Euler circuit if and only if n is even (and greater than 0, of course).

---

27. For which values of n do the graphs in Exercise 26 have an Euler path but no Euler circuit?

27. a) Clearly K_2 has an Euler path but no Euler circuit. For odd $n > 2$ there is an Euler circuit (since the degrees of all the vertices are $n-1$, which is even), whereas for even $n > 2$ there are at least 4 vertices of odd degree and hence no Euler path. Thus for no n other than 2 is there an Euler path but not an Euler circuit.

b) Since C_n has an Euler circuit for all n, there are no values of n meeting these conditions.

c) A wheel has at least 3 vertices of degree 3 (around the rim), so there can be no Euler path.

d) The same argument applies here as applied in part **(a)**. In more detail, Q_1 (which is the same as K_2) is the only cube with an Euler path but no Euler circuit, since for odd $n > 1$ there are too many vertices of odd degree, and for even $n > 1$ there is an Euler circuit.

---

28. For which values of m and n does the complete bipartite graph $K_{m,n}$ have an

a) Euler circuit?

b) Euler path?

28. a) Since the degrees of the vertices are all m and n, this graph has an Euler circuit if and only if both of the positive integers m and n are even.

b) All the graphs listed in part (a) have an Euler circuit, which is also an Euler path. In addition, the graphs $K_{2,n}$ for odd n (and $K_{m,2}$ for odd m) have exactly 2 vertices of odd degree, so they have an Euler path but not an Euler circuit. Also, $K_{1,1}$ obviously has an Euler path. All other complete bipartite graphs have too many vertices of odd degree.

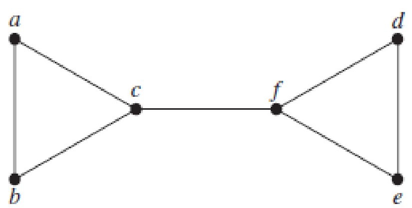
---

29. Find the least number of times it is necessary to lift a pencil from the paper when drawing each of the graphs in Exercises 1–7 without retracing any part of the graph.

29. Just as a graph with 2 vertices of odd degree can be drawn with one continuous motion, a graph with $2m$ vertices of odd degree can be drawn with m continuous motions. The graph in Exercise 1 has 4 vertices of odd degree, so it takes 2 continuous motions; in other words, the pencil must be lifted once. We could do this, for example, by first tracing a, c, d, e, a, b and then tracing c, b, e. The graphs in Exercises 2–7 all have Euler paths, so no lifting is necessary.

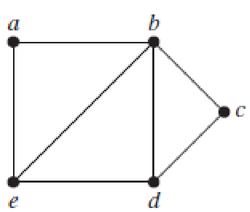
In Exercises 30–36 determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

30.

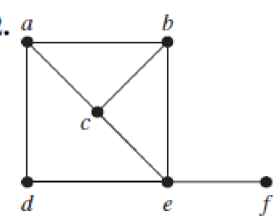


30. This graph can have no Hamilton circuit because of the cut edge $\{c, f\}$. Every simple circuit must be confined to one of the two components obtained by deleting this edge.

31.



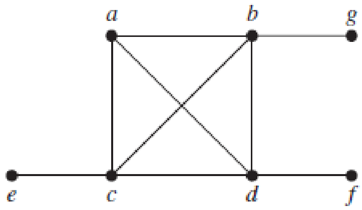
32.



31. It is clear that a, b, c, d, e, a is a Hamilton circuit.

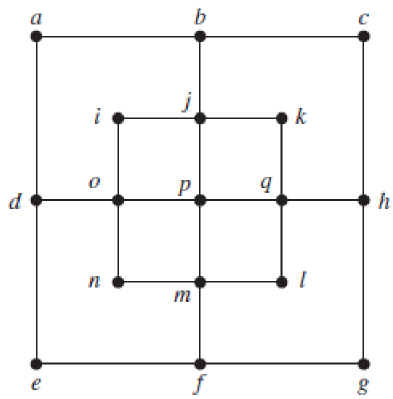
32. As in Exercise 30, the cut edge ($\{e, f\}$ in this case) prevents a Hamilton circuit.

33.



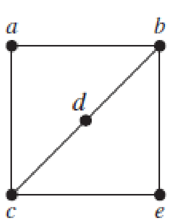
33. There is no Hamilton circuit because of the cut edges ($\{c, e\}$, for instance). Once a purported circuit had reached vertex e, there would be nowhere for it to go.

34.

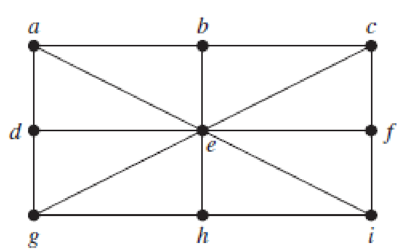


34. This graph has no Hamilton circuit. If it did, then certainly the circuit would have to contain edges $\{d, a\}$ and $\{a, b\}$, since these are the only edges incident to vertex a. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These eight edges already complete a circuit, and this circuit omits the nine vertices on the inside. Therefore there is no Hamilton circuit.

35.



36.



35. There is no Hamiltonian circuit in this graph. If there were one, then it would have to include all the edges of the graph, because it would have to enter and exit vertex a, enter and exit vertex d, and enter and exit vertex e. But then vertex c would have been visited more than once, a contradiction.

36. It is easy to find a Hamilton circuit here, such as a, d, g, h, i, f, c, e, b, and back to a.

37. Does the graph in Exercise 30 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

37. This graph has the Hamilton path a, b, c, f, d, e. This simple path hits each vertex once.

---

38. Does the graph in Exercise 31 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

38. This graph has the Hamilton path a, b, c, d, e.

---

39. Does the graph in Exercise 32 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

39. This graph has the Hamilton path f, e, d, a, b, c.

---

40. Does the graph in Exercise 33 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

40. This graph has no Hamilton path. There are three vertices of degree 1; each of them would have to be an end vertex of every Hamilton path. Since a path has only 2 ends, this is impossible.

***41.** Does the graph in Exercise 34 have a Hamilton path? If so, find such a path. If it does not, give an argument to show why no such path exists.

41. There are eight vertices of degree 2 in this graph. Only two of them can be the end vertices of a Hamilton path, so for each of the other six their two incident edges must be present in the path. Now if either all four of the "outside" vertices of degree 2 (a, c, g, and e) or all four of the "inside" vertices of degree 2 (i, k,

l, and n) are not end vertices, then a circuit will be completed that does not include all the vertices—either the outside square or the middle square. Therefore if there is to be a Hamilton path then exactly one of the inside corner vertices must be an end vertex, and each of the other inside corner vertices must have its two incident edges in the path. Without loss of generality we can assume that vertex i is an end, and that the path begins i, o, n, m, l, q, k, j. At this point, either the path must visit vertex p, in which case it gets stuck, or else it must visit b, in which case it will never be able to reach p. Either case gives a contradiction, so there is no Hamilton path.

---

44. For which values of n do the graphs in Exercise 26 have a Hamilton circuit?

44. a) Obviously K_n has a Hamilton circuit for all $n \geq 3$ but not for $n \leq 2$.

b) Obviously C_n has a Hamilton circuit for all $n \geq 3$.

c) A Hamilton circuit for C_n can easily be extended to one for W_n by replacing one edge along the rim of the wheel by two edges, one going to the center and the other leading from the center. Therefore W_n has a Hamilton circuit for all $n \geq 3$.

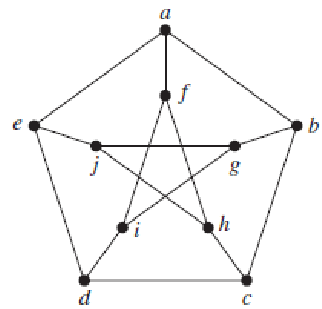
d) This is Exercise 49; see the solution given for it.

---

45. For which values of m and n does the complete bipartite graph $K_{m,n}$ have a Hamilton circuit?

45. A Hamilton circuit in a bipartite graph must visit the vertices in the parts alternately, returning to the part in which it began. Therefore a necessary condition is certainly $m = n$. Furthermore $K_{1,1}$ does not have a Hamilton circuit, so we need $n \geq 2$ as well. On the other hand, since the complete bipartite graph has all the edges we need, these conditions are sufficient. Explicitly, if the vertices are $a_1, a_2, \ldots, a_n$ in one part and $b_1, b_2, \ldots, b_n$ in the other, with $n \geq 2$, then one Hamilton circuit is $a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_1$.

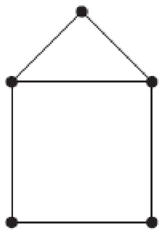
***46.** Show that the **Petersen graph**, shown here, does not have a Hamilton circuit, but that the subgraph obtained by deleting a vertex v, and all edges incident with v, does have a Hamilton circuit.



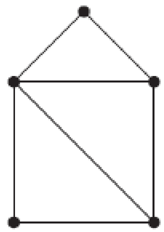
46. We do the easy part first, showing that the graph obtained by deleting a vertex from the Petersen graph has a Hamilton circuit. By symmetry, it makes no difference which vertex we delete, so assume that it is vertex j. Then a Hamilton circuit in what remains is $a, e, d, i, g, b, c, h, f, a$. Now we show that the entire graph has no Hamilton circuit. Assume that a Hamilton circuit exists. Not all the edges around the outside can be used, so without loss of generality assume that $\{c, d\}$ is not used. Then $\{e, d\}$, $\{d, i\}$, $\{h, c\}$, and $\{b, c\}$ must all be used. If $\{a, f\}$ is not used, then $\{e, a\}$, $\{a, b\}$, $\{f, i\}$, and $\{f, h\}$ must be used, forming a premature circuit. Therefore $\{a, f\}$ is used. Without loss of generality we may assume that $\{e, a\}$ is also used, and $\{a, b\}$ is not used. Then $\{b, g\}$ is also used, and $\{e, j\}$ is not. But this requires $\{g, j\}$ and $\{h, j\}$ to be used, forming a premature circuit b, c, h, j, g, b. Hence no Hamilton circuit can exist in this graph.

47. For each of these graphs, determine (*i*) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit, (*ii*) whether Ore's theorem can be used to show that the graph has a Hamilton circuit, and (*iii*) whether the graph has a Hamilton circuit.

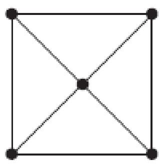
a)



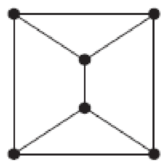
b)



c)



d)



47. For Dirac's theorem to be applicable, we need every vertex to have degree at least $n/2$, where n is the number of vertices in the graph. For Ore's theorem, we need $\deg(x)+\deg(y) \geq n$ whenever x and y are not adjacent.

a) In this graph $n = 5$. Dirac's theorem does not apply, since there is a vertex of degree 2, and 2 is smaller than $n/2$. Ore's theorem also does not apply, since there are two nonadjacent vertices of degree 2, so the sum of their degrees is less than n. However, the graph does have a Hamilton circuit—just go around the pentagon. This illustrates that neither of the sufficient conditions for the existence of a Hamilton circuit given in these theorems is necessary.

b) Everything said in the solution to part **(a)** is valid here as well.

c) In this graph $n = 5$, and all the vertex degrees are either 3 or 4, both of which are at least $n/2$. Therefore Dirac's theorem guarantees the existence of a Hamilton circuit. Ore's theorem must apply as well, since $(n/2) + (n/2) = n$; in this case, the sum of the degrees of any pair of nonadjacent vertices (there are only two such pairs) is 6, which is greater than or equal to 5.

d) In this graph $n = 6$, and all the vertex degrees are 3, which is (at least) $n/2$. Therefore Dirac's theorem guarantees the existence of a Hamilton circuit. Ore's theorem must apply as well, since $(n/2) + (n/2) = n$; in this case, the sum of the degrees of any pair of nonadjacent vertices is 6.

Although not illustrated in any of the examples in this exercise, there are graphs for which Ore's theorem applies, even though Dirac's does not. Here is one: Take K_4, and then tack on a path of length 2 between two of the vertices, say a, b, c. In all, this graph has five vertices, two with degree 3, two with degree 4, and one with degree 2. Since there is a vertex with degree less than $5/2$, Dirac's theorem does not apply. However, the sum of the degrees of any two (nonadjacent) vertices is at least $2 + 3 = 5$, so Ore's theorem does apply and guarantees that there is a Hamilton circuit.

SECTION 10.7 Planar Graphs

As with Euler and Hamilton circuits and paths, the topic of planar graphs is a classical one in graph theory. The theory (Euler's formula, Kuratowski's theorem, and their corollaries) is quite beautiful. It is easy to ask extremely difficult questions in this area, however—see Exercise 27, for example. In practice, there are very efficient algorithms for determining planarity that have nothing to do with Kuratowski's theorem, but they are quite complicated and beyond the scope of this book. For the exercises here, the best way to show that a graph is planar is to draw a planar embedding; the best way to show that a graph is nonplanar is to find a subgraph homeomorphic to K_5 or $K_{3,3}$. (Usually it will be $K_{3,3}$.)

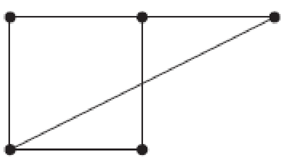
---

1. Can five houses be connected to two utilities without connections crossing?

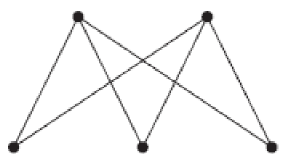
1. The question is whether $K_{5,2}$ is planar. It clearly is so, since we can draw it in the xy-plane by placing the five vertices in one part along the x-axis and the other two vertices on the positive and negative y-axis.

In Exercises 2–4 draw the given planar graph without any crossings.

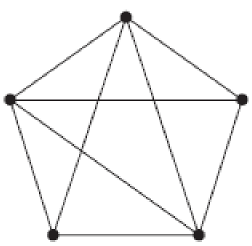
2.



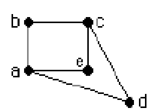
3.



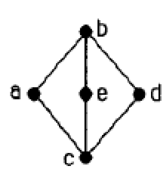
4.



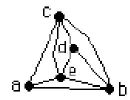
2. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex d down, then the crossings can be avoided.



3. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. This graph is just $K_{2,3}$; the picture below shows it redrawn by moving vertex c down.

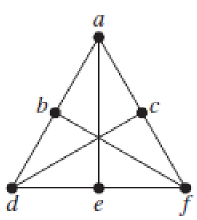


4. For convenience we label the vertices a, b, c, d, e, starting with the vertex in the lower left corner and proceeding clockwise around the outside of the figure as drawn in the exercise. If we move vertex b far to the right, and squeeze vertices d and e in a little, then we can avoid crossings.

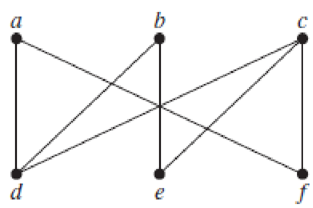


In Exercises 5–9 determine whether the given graph is planar. If so, draw it so that no edges cross.

5.

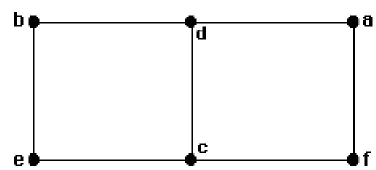


6.

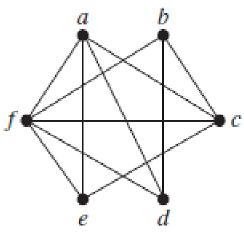


5. This is $K_{3,3}$, with parts $\{a, d, f\}$ and $\{b, c, e\}$. Therefore it is not planar.

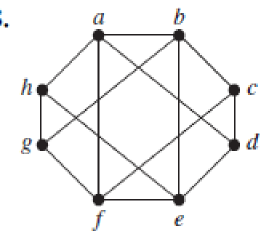
6. This graph is easily untangled and drawn in the following planar representation.



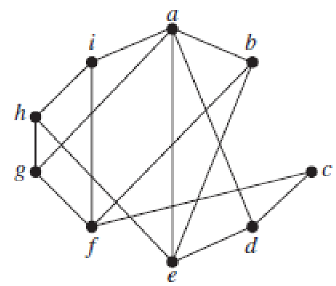
7.



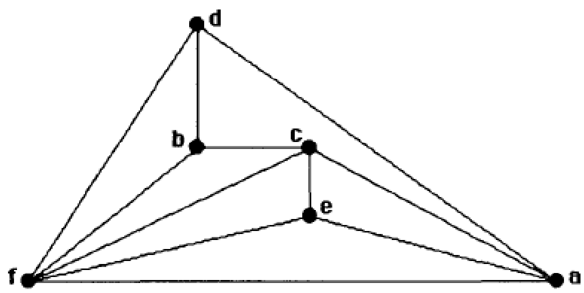
8.



9.



7. This graph can be untangled if we play with it long enough. The following picture gives a planar representation of it.



8. If one has access to software such as *The Geometer's Sketchpad*, then this problem can be solved by drawing the graph and moving the points around, trying to find a planar drawing. If we are unable to find one, then we look for a reason why—either a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$ (always try the latter first). In this case we find that there is in fact an actual copy of $K_{3,3}$, with vertices a, c, and e in one set and b, d, and f in the other.

9. If one has access to software such as *The Geometer's Sketchpad*, then this problem can be solved by drawing the graph and moving the points around, trying to find a planar drawing. If we are unable to find one, then we look for a reason why—either a subgraph homeomorphic to K_5 or one homeomorphic to $K_{3,3}$ (always try the latter first). In this case we find that there is a homeomorphic copy of $K_{3,3}$, with vertices b, g, and i in one set and a, f, and h in the other; all the edges are there except for the edge bh, and it is represented by the path beh.

---

10. Complete the argument in Example 3.

10. The argument is similar to the argument when v_3 is inside region R_2. In the case at hand the edges between v_3 and v_4 and between v_3 and v_5 separate R_1 into two subregions, R_{11} (bounded by v_1, v_4, v_3, and v_5) and R_{12} (bounded by v_2, v_4, v_3, and v_5). Now again there is no way to place vertex v_6 without forcing a crossing. If v_6 is in R_2, then there is no way to draw the edge $\{v_3, v_6\}$ without crossing another edge. If v_6 is in R_{11}, then the edge between v_2 and v_6 cannot be drawn; whereas if v_6 is in R_{12}, then the edge between v_1 and v_6 cannot be drawn.

11. Show that K_5 is nonplanar using an argument similar to that given in Example 3.

11. We give a proof by contradiction. Suppose that there is a planar representation of K_5, and let us call the vertices v_1, v_2, $\ldots$, v_5. There must be an edge from every vertex to every other. In particular, v_1, v_2, v_3, v_4, v_5, v_1 must form a pentagon. The pentagon separates the plane into two regions, an inside and an outside. The edge from v_1 to v_3 must be present, and without loss of generality let us assume it is drawn on the inside. Then there is no way for edges $\{v_2, v_4\}$ and $\{v_2, v_5\}$ to be in the inside, so they must be in the outside region. Now this prevents edges $\{v_1, v_4\}$ and $\{v_3, v_5\}$ from being on the outside. But they cannot both be on the inside without crossing. Therefore there is no planar representation of K_5.

---

12. Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

12. Euler's formula says that $v - e + r = 2$. We are given $v = 8$, and from the fact that the sum of the degrees equals twice the number of edges, we deduce that $e = (3 \cdot 8)/2 = 12$. Therefore $r = 2 - v + e = 2 - 8 + 12 = 6$.

---

13. Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

13. We apply Euler's formula $r = e - v + 2$. Here we are told that $v = 6$. We are also told that each vertex has degree 4, so that the sum of the degrees is 24. Therefore by the handshaking theorem there are 12 edges, so $e = 12$. Solving, we find $r = 8$.

---

14. Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

14. Euler's formula says that $v - e + r = 2$. We are given $e = 30$ and $r = 20$. Therefore $v = 2 - r + e = 2 - 20 + 30 = 12$.

---

15. Prove Corollary 3.

15. The proof is very similar to the proof of Corollary 1. First note that the degree of each region is at least 4. The reason for this is that there are no loops or multiple edges (which would give regions of degree 1 or 2) and no simple circuits of length 3 (which would give regions of degree 3); and the degree of the unbounded region is at least 4 since we are assuming that $v \geq 3$. Therefore we have, arguing as in the proof of Corollary 1, that $2e \geq 4r$, or simply $r \leq e/2$. Plugging this into Euler's formula, we obtain $e - v + 2 \leq e/2$, which gives $e \leq 2v - 4$ after some trivial algebra.

---

16. Suppose that a connected bipartite planar simple graph has e edges and v vertices. Show that $e \leq 2v - 4$ if $v \geq 3$.

16. A bipartite simple graph has no simple circuits of length three. Therefore the inequality follows from Corollary 3.

***17.** Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

17. The proof is exactly the same as in Exercise 15, except that this time the degree of each region must be at least 5. Thus we get $2e \geq 5r$, which after the same algebra as before, gives the desired inequality.

---

18. Suppose that a planar graph has k connected components, e edges, and v vertices. Also suppose that the plane is divided into r regions by a planar representation of the graph. Find a formula for r in terms of e, v, and k.

18. If we add $k-1$ edges, we can make the graph connected, create no new regions, and still avoid edge crossings. (We just add an edge from one vertex in one component, incident to the unbounded region, to one vertex in each of the other components.) For this new graph, Euler's formula tells us that $v - (e + k - 1) + r = 2$. This simplifies algebraically to $r = e - v + k + 1$.

---

19. Which of these nonplanar graphs have the property that the removal of any vertex and all edges incident with that vertex produces a planar graph?

a) K_5 **b)** K_6 **c)** $K_{3,3}$ **d)** $K_{3,4}$

19. a) If we remove a vertex from K_5, then we get K_4, which is clearly planar.

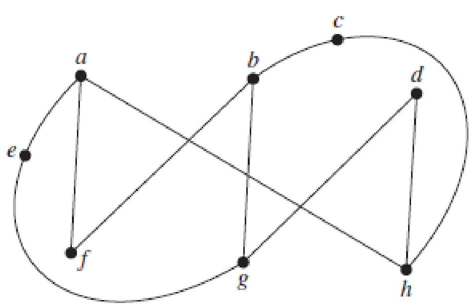
b) If we remove a vertex from K_6, then we get K_5, which is not planar.

c) If we remove a vertex from $K_{3,3}$, then we get $K_{3,2}$, which is clearly planar.

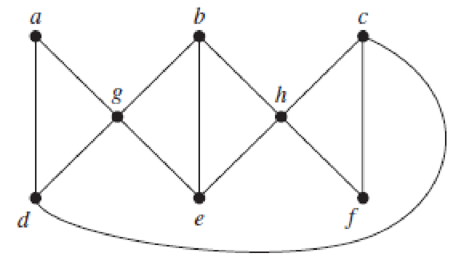
d) We assume the question means "Is it the case that for every v, the removal of v makes the graph planar?" Then the answer is no, since we can remove a vertex in the part of size 4 to leave $K_{3,3}$, which is not planar.

In Exercises 20–22 determine whether the given graph is homeomorphic to $K_{3,3}$.

20.



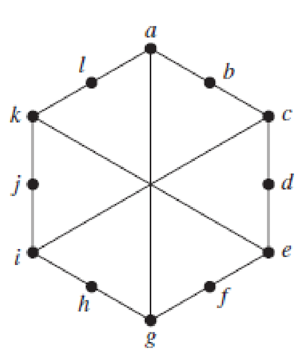
21.



20. This graph is not homeomorphic to $K_{3,3}$, since by rerouting the edge between a and h we see that it is planar.

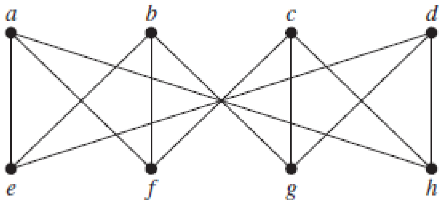
21. This graph is planar and hence cannot be homeomorphic to $K_{3,3}$.

22.

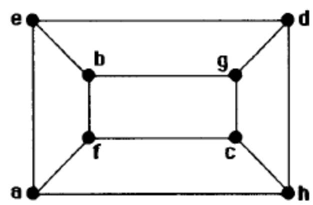


22. Replace each vertex of degree two and its incident edges by a single edge. Then the result is $K_{3,3}$: the parts are $\{a, e, i\}$ and $\{c, g, k\}$. Therefore this graph is homeomorphic to $K_{3,3}$.

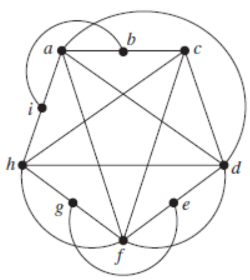
In Exercises 23–25 use Kuratowski's theorem to determine whether the given graph is planar.

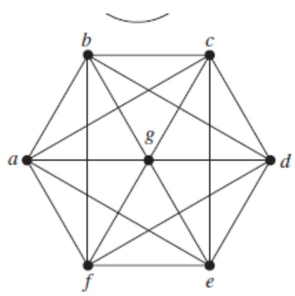
23. 

23. The instructions are really not fair. It is hopeless to try to use Kuratowski's theorem to prove that a graph *is* planar, since we would have to check hundreds of cases to argue that there is no subgraph homeomorphic to K_5 or $K_{3,3}$. Thus we will show that this graph is planar simply by giving a planar representation. Note that it is Q_3.



---

24. 

25. 

24. This graph is nonplanar. If we delete the five curved edges outside the big pentagon, then the graph is homeomorphic to K_5. We can see this by replacing each vertex of degree 2 and its two edges by one edge.

25. This graph is nonplanar, since it contains $K_{3,3}$ as a subgraph: the parts are $\{a, g, d\}$ and $\{b, c, e\}$. (Actually it contains $K_{3,4}$, and it even contains a subgraph homeomorphic to K_5.)

---

26. Show that $K_{3,3}$ has 1 as its crossing number.

26. If we follow the proof in Example 3, we see how to construct a planar representation of all of $K_{3,3}$ except for one edge. In particular, if we place vertex v_6 inside region R_{22} of Figure 7(b), then we can draw edges from v_6 to v_2 and v_3 with no crossings, and to v_1 with only one crossing. Furthermore, since $K_{3,3}$ is not planar, its crossing number cannot be 0. Hence its crossing number is 1.

**27. Find the crossing numbers of each of these nonplanar graphs.

a) K_5 b) K_6 c) K_7

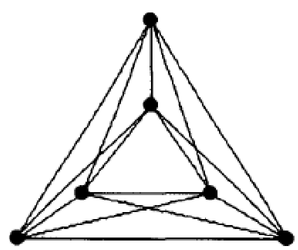
d) $K_{3,4}$ e) $K_{4,4}$ f) $K_{5,5}$

27. This is an extremely hard problem. We will present parts of the solution; the reader should consult a good graph theory book, such as Gary Chartrand, Linda Lesniak and Ping Zhang's *Graphs & Digraphs*, fifth edition (Chapman & Hall/CRC Press, 2011), for references and further details.

First we will state, without proof, what is known about crossing numbers for complete graphs (much is still not known about crossing numbers). If $n \leq 10$, then the crossing number of K_n is given by the following product

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

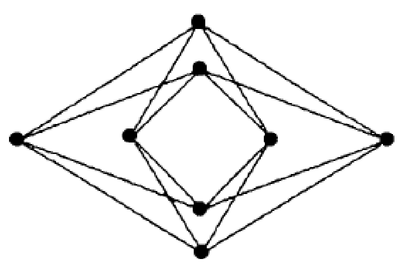
Thus the answers for parts **(a)**, **(b)**, and **(c)** are 1, 3, and 9, respectively. The figure below shows K_6 drawn in the plane with three crossings, which at least proves that the crossing number of K_6 is at most 3. The proof that it is not less than **3** is not easy. The embedding of K_5 with one crossing can be seen in this same picture, by ignoring the vertex at the top.



Second, for the complete bipartite graphs, what is known is that if the smaller of m and n is at most 6, then the crossing number of $K_{m,n}$ is given by the following product

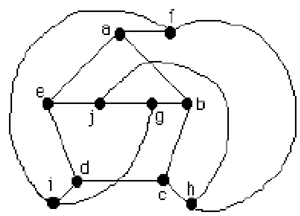
$$\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor.$$

Thus the answers for parts **(d)**, **(e)**, and **(f)** are 2, 4, and 16, respectively. The figure below shows $K_{4,4}$ drawn in the plane with four crossings, which at least proves that the crossing number of $K_{4,4}$ is at most 4. The proof that it is not less than 4 is, again, difficult. It is also easy to see from this picture that the crossing number of $K_{3,4}$ is at most 2 (by ignoring the top vertex).



***28.** Find the crossing number of the Petersen graph.

28. First note that the Petersen graph with one edge removed is not planar; indeed, by Example 9, the Petersen graph with three mutually adjacent edges removed is not planar. Therefore the crossing number must be greater than 1. (If it were only 1, then removing the edge that crossed would give a planar drawing of the Petersen graph minus one edge.) The following figure shows a drawing with only two crossings. (This drawing was obtained by a little trial and error.) Therefore the crossing number must be 2. (In this figure, the vertices are labeled as in Figure 14(a).)

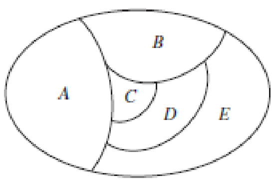


SECTION 10.8 Graph Coloring

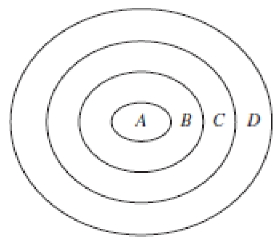
Like the problem of finding Hamilton paths, the problem of finding colorings with the fewest possible colors probably has no good algorithm for its solution. In working these exercises, for the most part you should proceed by trial and error, using whatever insight you can gain by staring at the graph (for instance, finding large complete subgraphs). There are also some interesting exercises here on coloring the edges of graphs—see Exercises 21–26. Exercises 29–31 are worth looking at, as well: they deal with a fast algorithm for coloring a graph that is not guaranteed to produce an optimal coloring.

In Exercises 1–4 construct the dual graph for the map shown. Then find the number of colors needed to color the map so that no two adjacent regions have the same color.

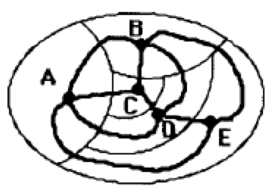
1.



2.

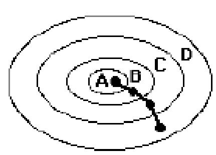


1. We construct the dual graph by putting a vertex inside each region (but not in the unbounded region), and drawing an edge between two vertices if the regions share a common border. The easiest way to do this is illustrated in our answer. First we draw the map, then we put a vertex inside each region and make the connections. The dual graph, then, is the graph with heavy lines.



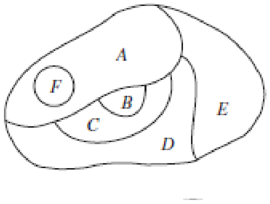
The number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Since A, B, C, and D are mutually adjacent, at least four colors are needed. We can color each of the vertices (i.e., regions) A, B, C, and D a different color, and we can give E the same color as we give C.

2. We construct the dual as in Exercise 1.

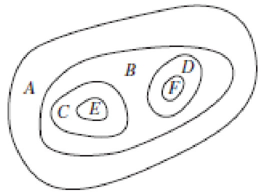


As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A and C, and the other for B and D.

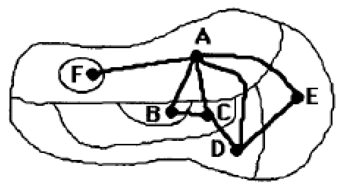
3.



4.

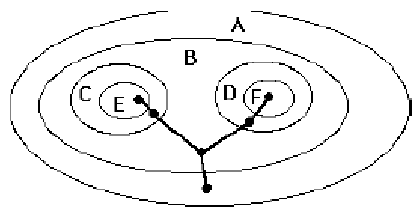


3. We construct the dual as in Exercise 1.



As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Three colors are clearly necessary, because of the triangle ABC, for instance. Furthermore three colors suffice, since we can color vertex (region) A red, vertices B, D, and F blue, and vertices C and E green.

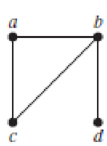
4. We construct the dual as in Exercise 1.



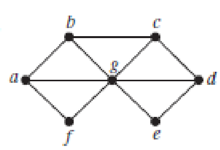
As in Exercise 1, the number of colors needed to color this map is the same as the number of colors needed to color the dual graph. Clearly two colors are necessary and sufficient: one for vertices (regions) A, C, and D, and the other for B, E, and F.

In Exercises 5–11 find the chromatic number of the given graph.

5.



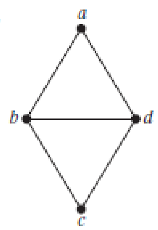
6.



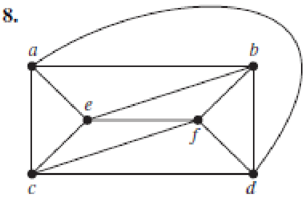
5. For Exercises 5–11, in order to prove that the chromatic number is k, we need to find a k-coloring and to show that (at least) k colors are needed. Here, since there is a triangle, at least 3 colors are needed. Clearly 3 colors suffice, since we can color a and d the same color.

6. Since there is a triangle, at least 3 colors are needed. To show that 3 colors suffice, notice that we can color the vertices around the outside alternately using red and blue, and color vertex g green.

7.



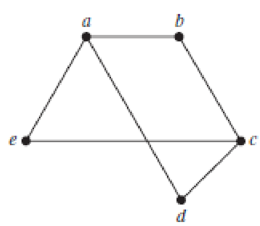
8.



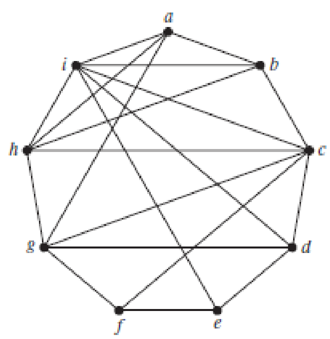
7. Since there is a triangle, at least 3 colors are needed. Clearly 3 colors suffice, since we can color a and c the same color.

8. Since there is a triangle, at least 3 colors are needed. The coloring in which b and c are blue, a and f are red, and d and e are green shows that 3 colors suffice.

9.



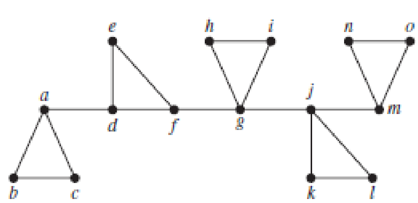
10.



9. Since there is an edge, at least 2 colors are needed. The coloring in which b, d, and e are red and a and c blue shows that 2 colors suffice.

10. Since vertices b, c, h, and i form a K_4, at least 4 colors are required. A coloring using only 4 colors (and we can get this by trial and error, without much difficulty) is to let a and c be red; b, d, and f, blue; g and i, green; and e and h, yellow.

11.



11. Since there is a triangle, at least 3 colors are needed. It is not hard to construct a 3-coloring. We can let a, f, h, j, and n be blue; let b, d, g, k, and m be green; and let c, e, i, l, and o be yellow.

12. For the graphs in Exercises 5–11, decide whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.

12. In Exercise 5 the chromatic number is 3, but if we remove vertex a, then the chromatic number will fall to 2. In Exercise 6 the chromatic number is 3, but if we remove vertex g, then the chromatic number will fall to 2. In Exercise 7 the chromatic number is 3, but if we remove vertex b, then the chromatic number will fall to 2. In Exercise 8 the chromatic number was shown to be 3. Even if we remove a vertex, at least one of the two triangles ace and bdf must remain, since they share no vertices. Therefore the smaller graph will still have chromatic number 3. In Exercise 9 the chromatic number is 2. Obviously it is not possible to reduce it to 1 by removing one vertex, since at least one edge will remain. In Exercise 10 the chromatic number was shown to be 4, and a coloring was provided. If we remove vertex h and recolor vertex e red, then we can eliminate color yellow from that solution. Therefore we will have reduced the chromatic number to 3. Finally, the graph in Exercise 11 will still have a triangle, no matter what vertex is removed, so we cannot lower its chromatic number below 3 by removing a vertex.

---

13. Which graphs have a chromatic number of 1?

13. If a graph has an edge (not a loop, since we are assuming that the graphs in this section are simple), then its chromatic number is at least 2. Conversely, if there are no edges, then the coloring in which every vertex receives the same color is proper. Therefore a graph has chromatic number 1 if and only if it has no edges.

---

14. What is the least number of colors needed to color a map of the United States? Do not consider adjacent states that meet only at a corner. Suppose that Michigan is one region. Consider the vertices representing Alaska and Hawaii as isolated vertices.

14. Since the map is planar, we know that four colors suffice. That four colors are necessary can be seen by looking at Kentucky. It is surrounded by Tennessee, Missouri, Illinois, Indiana, Ohio, West Virginia, and Virginia; furthermore the states in this list form a C_7, each one adjacent to the next. Therefore at least three colors are needed to color these seven states (see Exercise 16), and then a fourth is necessary for Kentucky.

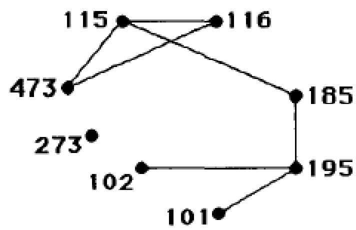
---

15. What is the chromatic number of W_n?

15. In Example 4 we saw that the chromatic number of C_n is 2 if n is even and 3 if n is odd. Since the wheel W_n is just C_n with one more vertex, adjacent to all the vertices of the C_n along the rim of the wheel, W_n clearly needs exactly one more color than C_n (for that middle vertex). Therefore the chromatic number of W_n is 3 if n is even and 4 if n is odd.

17. Schedule the final exams for Math 115, Math 116, Math 185, Math 195, CS 101, CS 102, CS 273, and CS 473, using the fewest number of different time slots, if there are no students taking both Math 115 and CS 473, both Math 116 and CS 473, both Math 195 and CS 101, both Math 195 and CS 102, both Math 115 and Math 116, both Math 115 and Math 185, and both Math 185 and Math 195, but there are students in every other pair of courses.

17. Consider the graph representing this problem. The vertices are the 8 courses, and two courses are joined by an edge if there are students taking both of them. Thus there are edges between every pair of vertices except the 7 pairs listed. It is much easier to draw the complement than to draw this graph itself; it is shown below.

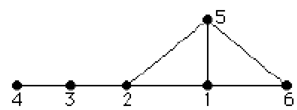


We want to find the chromatic number of the graph whose complement we have drawn; the colors will be the time periods for the exams. First note that since Math 185 and the four CS courses form a K_5 (in other words, there are no edges between any two of these in our picture), the chromatic number is at least 5. To show that it equals 5, we just need to color the other three vertices. A little trial and error shows that we can make Math 195 the same color as (i.e., have its final exam at the same time as) CS 101; and we can make Math 115 and 116 the same color as CS 473. Therefore five time slots (colors) are sufficient.

18. How many different channels are needed for six stations located at the distances shown in the table, if two stations cannot use the same channel when they are within 150 miles of each other?

	1	*2*	*3*	*4*	*5*	*6*
1	—	85	175	200	50	100
2	85	—	125	175	100	160
3	175	125	—	100	200	250
4	200	175	100	—	210	220
5	50	100	200	210	—	100
6	100	160	250	220	100	—

18. We draw the graph in which two vertices (representing locations) are adjacent if the locations are within 150 miles of each other.



Clearly three colors are necessary and sufficient to color this graph, say red for vertices 4, 2, and 6; blue for 3 and 5; and yellow for 1. Thus three channels are necessary and sufficient.

21. Find the edge chromatic number of each of the graphs in Exercises 5–11.

21. Note that the number of colors needed to color the edges is at least as large as the largest degree of a vertex, since the edges at each vertex must all be colored differently. Hence if we can find an edge coloring with that many colors, then we know we have found the answer. In Exercise 5 there is a vertex of degree 3, so the edge chromatic number is at least 3. On the other hand, we can color $\{a, c\}$ and $\{b, d\}$ the same color, so 3 colors suffice. In Exercise 6 the 6 edges incident to g must all get different colors. On the other hand, it is not hard to complete a proper edge coloring with only these colors (for example, color edge $\{a, f\}$ with the same color as used on $\{d, g\}$), so the answer is 6. In Exercise 7 the answer must be at least 3; it is 3 since edges that appear as parallel line segments in the picture can have the same color. In Exercise 8 clearly 4 colors are required, since the vertices have degree 4. In fact 4 colors are sufficient. Here is one proper 4-coloring (we denote edges in the obvious shorthand notation): color 1 for ac, be, and df; color 2 for ae, bd, and cf; color 3 for ab, cd, and ef; and color 4 for ad, bf, and ce. In Exercise 9 the answer must be at least 3; it is easy to construct a 3-coloring of the edges by inspection: $\{a, b\}$ and $\{c, e\}$ have the same color, $\{a, d\}$ and $\{b, c\}$ have the same color, and $\{a, e\}$ and $\{c, d\}$ have the same color. In Exercise 10 the largest degree is 6 (vertex i has degree 6); therefore at least 6 colors are required. By trial and error we come up with this coloring using 6 colors (we use the obvious shorthand notation for edges); there are many others, of course. Assign color 1 to ag, cd, and hi; color 2 to ab, cf, dg, and ei; color 3 to bh, cg, di, and ef; color 4 to ah, ci, and de; color 5 to bi, ch, and fg; and color 6 to ai, bc, and gh. Finally, in Exercise 11 it is easy to construct an edge-coloring with 4 colors; again the edge chromatic number is the maximum degree of a vertex.

Despite the appearances of these examples, it is not the case that the edge chromatic number of a graph is always equal to the maximum degree of the vertices in the graph. The simplest example in which this is not true is K_3. Clearly its edge chromatic number is 3 (since all three edges are adjacent to each other), but its maximum degree is 2. There is a theorem, however, stating that the edge chromatic number is always equal to either the maximum degree or one more than the maximum degree.

---

23. Find the edge chromatic numbers of

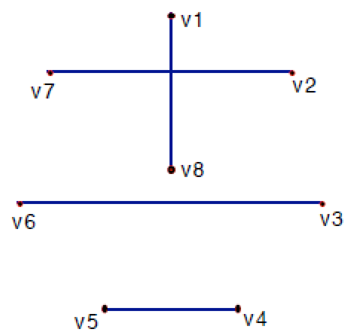
a) C_n, where $n \geq 3$.

b) W_n, where $n \geq 3$.

23. **a)** The n-cycle's edges are just like the n-cycle's vertices (each adjacent to the next as we go around the cycle), so the edge chromatic number is the same, namely 2 if n is even and 3 if n is odd, as in Example 4.
b) The edge chromatic number is at least n, because the radial edges are all pairwise adjacent and therefore must all have distinct colors. Suppose we call these colors 1 through n proceeding clockwise. We need no additional colors for the edges of the cycle, because we can color the edge adjacent to the spokes colored 1 and 2 with color 3 and proceed clockwise with colors $4, 5, \ldots, n-1, n, 1$, and 2. Therefore $\chi'(W_n) = n$.

***26.** Find the edge chromatic number of K_n when n is a positive integer.

26. This is really a problem about scheduling a round-robin tournament. Let the vertices of K_n be $v_1, v_2, \ldots, v_n$. These are the players in the tournament. We join two vertices with an edge of color i if those two players meet in round i of the tournament. First suppose that n is even. Place v_n in the center of a circle, with the remaining vertices evenly spaced on the circle, as shown here for $n = 8$. The first round of the tournament uses edges $v_n v_1$, $v_2 v_{n-1}$, $v_3 v_{n-2}$, $\ldots$, $v_{n/2} v_{(n/2)+1}$; these edges, shown in the diagram, get color 1.



---

28. What can be said about the chromatic number of a graph that has K_n as a subgraph?

28. Since each of the n vertices in this subgraph must have a different color, the chromatic number must be at least n.

---

36. Find these values:

a) $\chi_2(K_3)$ **b)** $\chi_2(K_4)$ **c)** $\chi_2(W_4)$

d) $\chi_2(C_5)$ **e)** $\chi_2(K_{3,4})$ **f)** $\chi_3(K_5)$

***g)** $\chi_3(C_5)$ **h)** $\chi_3(K_{4,5})$

36. First let us prove some general results. In a complete graph, each vertex is adjacent to every other vertex, so each vertex must get its own set of k different colors. Therefore if there are n vertices, kn colors are clearly necessary and sufficient. Thus $\chi_k(K_n) = kn$. In a bipartite graph, every vertex in one part can get the same set of k colors, and every vertex in the other part can get the same set of k colors (a disjoint set from the colors assigned to the vertices in the first part). Therefore $2k$ colors are sufficient, and clearly $2k$ colors are required if there is at least one edge. Let us now look at the specific graphs.

a) For this complete graph situation we have $k = 2$ and $n = 3$, so $2 \cdot 3 = 6$ colors are necessary and sufficient.

b) As in part (a), the answer is kn, which here is $2 \cdot 4 = 8$.

c) Call the vertex in the middle of the wheel m, and call the vertices around the rim, in order, a, b, c, and d. Since m, a, and b form a triangle, we need at least 6 colors. Assign colors 1 and 2 to m, 3 and 4 to a, and 5 and 6 to b. Then we can also assign 3 and 4 to c, and 5 and 6 to d, completing a 2-tuple coloring with 6 colors. Therefore $\chi_2(W_4) = 6$.

d) First we show that 4 colors are not sufficient. If we had only colors 1 through 4, then as we went around the cycle we would have to assign, say, 1 and 2 to the first vertex, 3 and 4 to the second, 1 and 2 to the third, and 3 and 4 to the fourth. This gives us no colors for the final vertex. To see that 5 colors are sufficient, we simply give the coloring: In order around the cycle the colors are $\{1,2\}$, $\{3,4\}$, $\{1,5\}$, $\{2,4\}$, and $\{3,5\}$. Therefore $\chi_2(C_5) = 5$.

e) By our general result on bipartite graphs, the answer is $2k = 2 \cdot 2 = 4$.

f) By our general result on complete graphs, the answer is $kn = 3 \cdot 5 = 15$.

g) We claim that the answer is 8. To see that eight colors suffice, we can color the vertices as follows in order around the cycle: $\{1,2,3\}$, $\{4,5,6\}$, $\{1,2,7\}$, $\{3,6,8\}$, and $\{4,5,7\}$. Showing that seven colors are not sufficient is harder. Assume that a coloring with seven colors exists. Without loss of generality, color the first vertex $\{1,2,3\}$ and color the second vertex $\{4,5,6\}$. If the third vertex is colored $\{1,2,3\}$, then the fourth and fifth vertices would need to use six colors different from 1, 2, and 3, for a total of nine colors. Therefore

***37.** Let G and H be the graphs displayed in Figure 3. Find

a) $\chi_2(G)$. b) $\chi_2(H)$.
c) $\chi_3(G)$. d) $\chi_3(H)$.

37. a) Note that vertices d, e, and f are mutually adjacent. Therefore six different colors are needed in a 2-tuple coloring, since each of these three vertices needs a disjoint set of two colors. In fact it is easy to give a coloring with just six colors: Color a, d, and g with $\{1,2\}$; color c and e with $\{3,4\}$; and color b and f with $\{5,6\}$. Thus $\chi_2(G)=6$.

b) This one is trickier than part **(a)**. There is no coloring with just six colors, since if there were, we would be forced (without loss of generality) to color d with $\{1,2\}$; e with $\{3,4\}$; f with $\{5,6\}$; then g with $\{1,2\}$, b with $\{5,6\}$, and c with $\{3,4\}$. This gives no free colors for vertex a. Now this may make it appear that eight colors are required, but a little trial and error shows us that seven suffice: Color a with $\{2,4\}$; color b and f with $\{5,6\}$; color d with $\{1,2\}$; color c with $\{3,7\}$; color e with $\{3,4\}$; and color g with $\{1,7\}$. Thus $\chi_2(H)=7$.

c) This is similar to part **(a)**. Here nine colors are necessary and sufficient, since a, d, and g can get one set of three colors; b and f can get a second set; and c and e can get a third set. Clearly nine colors are necessary to color the triangles.

d) First we construct a coloring with 11 colors: Color a with $\{3,6,11\}$; color b and f with $\{7,8,9\}$; color d with $\{1,2,3\}$; color c with $\{4,5,10\}$; color e with $\{4,6,11\}$; and color g with $\{1,2,5\}$. To prove that $\chi_3(H)=11$, we must show that it is impossible to give a 3-tuple coloring with only ten colors. If such a coloring were possible, without loss of generality we could color d with $\{1,2,3\}$, e with $\{4,5,6\}$, f with $\{7,8,9\}$, and g with $\{1,2,10\}$. Now nine colors are needed for the three vertices a, b, and c, since they form a triangle; but colors 1 and 2 are already used in vertices adjacent to all three of them. Therefore at least $9+2=11$ colors are necessary.